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Exact solutions to a class of feedback systems on SO(n)[☆]

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ABSTRACT

This paper provides a novel approach to the problem of attitude tracking for a class of almost globally asymptotically stable feedback laws on SO(n). The closed-loop systems are solved exactly for the rotation matrices as explicit functions of time, the initial conditions, and the gain parameters of the control laws. The exact solutions provide insight into the transient dynamics of the system and can be used to prove almost global attractiveness of the identity matrix. Applications of these results are found in model predictive control problems where detailed insight into the transient attitude dynamics is utilized to approximately complete a task of secondary importance. Knowledge of the future trajectory of the states can also be used as an alternative to the zero-order hold in systems where the attitude is sampled at discrete time instances.

1. Introduction

The nonlinear control problem of stabilizing the attitude dynamics of a rigid body has a long history of study and is important in a diverse range of engineering applications related to *e.g.* quadrotors (Lee, Leoky, & McClamroch, 2010), inverted 3-D pendulums (Chaturvedi, McClamroch, & Bernstein, 2009), and robotic manipulators (Hu et al., 2009). It is interesting from a theoretical point of view due to the nonlinear state equations and the topology of the underlying state space SO(3). An often cited result states that global asymptotical stability on SO(3) cannot be achieved by means of a continuous, time-invariant feedback (Bhat & Bernstein, 2000). The literature does however provide results such as almost global asymptotical stability through continuous time-invariant feedback (Chaturvedi, Sanyal, & McClamroch, 2011; Sanyal, Fosbury, Chaturvedi, & Bernstein, 2009), almost semi-global stability (Lee, 2012), or global stability by means of a hybrid control approach (Mayhew, Sanfelice, & Teel, 2011b). The parameterizations used to represent SO(3) has important implications for the limits of control performance (Bhat & Bernstein, 2000; Chaturvedi et al., 2011; Mayhew, Sanfelice, & Teel, 2011a). In particular, the use of local representations yields local results. In most cases, it is preferable to either use global representations such as the unit quaternions or to work with the space of rotation matrices directly (Chaturvedi et al., 2011).

The exact solutions of a closed-loop system gives a detailed picture of both its transients and asymptotical behaviour and can hence be of use in control applications. The literature on solutions to attitude dynamics can be divided into two categories. Firstly, in a number of works the solutions are obtained during the control design process, *e.g.* using exact linearization (Dwyer, 1984) or optimal control design techniques such as the Pontryagin maximum principle (Spindler, 1998). Secondly, there are works whose main focus is solving the equations defining rigid-body dynamics under a set of specific assumptions (Ayoubi & Longuski, 2009; Doroshin, 2012; Elipe & Lanchares, 2008). This paper falls into the second category.

There is a considerable literature on the kinematics and dynamics of n-dimensional rigid-bodies. This literature includes works on attitude stabilization (Maithripala, Berg, & Dayawansa, 2006), attitude synchronization (Lageman, Sarlette, & Sepulchrev, 2009), distributed averaging (Matni & Horowitz, 2014), and generalized Newtonian equations of motion (Hurtado & Sinclair, 2004). It also includes the authors previous work (Markdahl & Hu, 2014; Markdahl, Thunberg, Hoppe, & Hu, 2013), which we shall comment on shortly. A key difference between the study of SO(3) and SO(n) is that parameterizations such as the unit quaternions cannot be used. Another is the motivation: work on SO(3) is usually motivated by applications concerning the attitude of rigid bodies. Work on SO(n) is not only of theoretical concern however, it also finds applications in the visualization of high-dimensional data (Thakur, 2008).

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The main contribution of this paper is to provide exact solutions to differential equations representing closed feedback loops on $\mathsf{SO}(n)$. Recent work on this problem include Markdahl, Hoppe, Wang, and Hu (2012), Markdahl et al. (2013) and Markdahl and Hu (2014). Other works such as Ayoubi and Longuski (2009), Doroshin (2012) and Elipe and Lanchares (2008) are related in spirit but address somewhat different problems. The work Markdahl et al. (2012) considers the solutions to closed-loop kinematics on $\mathsf{SO}(3)$. An application towards model predictive control (MPC) is proposed but left unexplored. The more general problem of solving two differential equations on $\mathsf{SO}(n)$ is treated in Markdahl et al. (2013). An application towards the problem of continuous actuation under discrete-time sensing is considered. The work Markdahl and Hu (2014) generalizes the results of Markdahl et al. (2013) to a greater class of feedback laws. This paper in turn generalizes Markdahl and Hu (2014) and explores the applications proposed in Markdahl et al. (2012, 2013). Note that many of the results of this paper easily extends to the case of $\mathsf{SE}(n)$ and may be combined with position control laws in an inner–outer loop feedback scheme to achieve pose stabilization on $\mathsf{SE}(n)$ (Roza & Maggiore, 2012).

This paper is structured as follows. Section 2 recalls the notation and some basic properties of matrix analysis, it can be skipped if the reader is familiar with that topic. Section 3 presents the attitude stabilization problem and introduces Problem 1, the problem of solving the closed-loop state equations. Section 4.1 generalizes a class of known control laws on $\mathsf{SO}(3)$ to the case of $\mathsf{SO}(n)$. It contains the main result of this paper, the solution to Problem 1. It also makes use of the exact solutions to prove that the proposed algorithms stabilize System 1 almost globally. Section 6 explores practical applications of the exact solutions to problems of model predictive control and continuous feedback in sampled systems. Section 8 provides some brief concluding remarks.

2. Preliminaries

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. The spectrum of $\mathbf{A}$ is written $\sigma(\mathbf{A})$. The transpose and conjugate transpose of $\mathbf{A}$ is written $\mathbf{A}^\top$ and $\mathbf{A}^*$ respectively. The commutator of $\mathbf{A}$ and $\mathbf{B}$ is defined by $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$. Their inner product is defined by $\langle \mathbf{A}, \mathbf{B}\rangle = \mathrm{tr}(\mathbf{A}^\top\mathbf{B})$ and the Frobenius norm by $\|\mathbf{A}\|_F = \langle \mathbf{A}, \mathbf{A}\rangle^{\frac{1}{2}}$.

The set of nonsingular matrices over a field $\mathcal{F}$ is denoted by $\mathsf{GL}(n, \mathcal{F})$. The unitary group is denoted by $\mathsf{U}(n) = \{\mathbf{U} \in \mathsf{GL}(n, \mathbb{C}) \mid \mathbf{U}^{-1} = \mathbf{U}^*\}$. The orthogonal group is $\mathsf{O}(n) = \{\mathbf{Q} \in \mathsf{GL}(n, \mathbb{R}) \mid \mathbf{Q}^{-1} = \mathbf{Q}^\top\}$. The special orthogonal group is denoted by $\mathsf{SO}(n) = \{\mathbf{R} \in \mathsf{O}(n) \mid \det \mathbf{R} = 1\}$. In this paper we define

$$\mathcal{R} = \{\mathbf{R} \in \mathsf{SO}(n) \mid -1 \in \sigma(\mathbf{R})\}.$$

It can be shown that $\{\mathbf{R} \in \mathsf{SO}(n) \mid \mathbf{R}^\top = \mathbf{R}\}/\{\mathbf{I}\} \subset \mathcal{R}$. Equality holds in the cases of $n \in \{2, 3\}$.

The Lie algebra of $\mathsf{SO}(n)$ is denoted by $\mathsf{so}(n) = \{\mathbf{S} \in \mathbb{R}^{n\times n} \mid \mathbf{S}^\top = -\mathbf{S}\}$. In this paper, we use $\mathbf{S}$ to denote the matrix $\mathrm{Log}\,\mathbf{R} \in \mathsf{so}(n)$ for $\mathbf{R} \in \mathsf{SO}(n) \setminus \mathcal{R}$.

The set of symmetric matrices is $\mathcal{S} = \{\mathbf{P} \in \mathbb{R}^{n\times n} \mid \mathbf{P}^\top = \mathbf{P}\}$. The set of positive-semidefinite matrices is denoted by $\mathcal{P} = \{\mathbf{P} \in \mathcal{S} \mid \sigma(\mathbf{P}) \subset [0, \infty)\}$. The set of positive-definite matrices is $\mathcal{P} \cap \mathsf{GL}(n, \mathbb{R})$.

The solution to a differential equation $\dot{\mathbf{X}} = \mathbf{F}(t, \mathbf{X})$ is denoted $\mathbf{X}(t; t_0, \mathbf{X}_0)$ where t is the time, t_0 is the initial time, and $\mathbf{X}_0$ is the initial condition. If the system is time independent we set $t_0 = 0$ and omit this dependence.

The principal matrix logarithm Log is defined on the set $\{\mathbf{A} \in \mathsf{GL}(n, \mathbb{R}) \mid \sigma(\mathbf{A}) \cap (-\infty, 0] = \varnothing\}$ (Culver, 1966). It satisfies $\mathrm{Im}\,\sigma(\mathrm{Log}\,\mathbf{A}) \subset \{z \in i\mathbb{R} \mid |z| < \pi\}$ (Higham, 2008). Since any rotation matrix $\mathbf{R}$ is normal, it follows that $\mathbf{R} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$ and the logarithm of $\mathbf{R}$ may be calculated as $\mathrm{Log}\,\mathbf{R} = \mathbf{U}\,\mathrm{Log}(\boldsymbol{\Lambda})\mathbf{U}^*$, where $\mathbf{U} \in \mathsf{U}(n)$. Moreover, $\boldsymbol{\Lambda} = \exp(i\boldsymbol{\Theta})$ for a diagonal matrix $\boldsymbol{\Theta}$ which satisfies $\boldsymbol{\Theta}_{ii} \in (-\pi, \pi)$ for all $\mathbf{R} \in \mathsf{SO}(n) \setminus \mathcal{R}$. Hence $\mathrm{Log}(\boldsymbol{\Lambda}) = i\boldsymbol{\Theta}$ and $\mathrm{Log}\,\mathbf{R} = i\mathbf{U}\boldsymbol{\Theta}\mathbf{U}^*$. The matrix logarithm allows us to calculate the geodesic distance between $\mathbf{R}_1, \mathbf{R}_2 \in \mathsf{SO}(n)$ using the Riemannian metric

$$d_R(\mathbf{R}_1, \mathbf{R}_2) = \frac{1}{\sqrt{2}}\|\mathrm{Log}(\mathbf{R}_1^\top\mathbf{R}_2)\|_F.$$

By the kth root of a normal matrix $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*$ we refer to its principal root, the normal matrix $\mathbf{A}^{\frac{1}{k}} = \mathbf{U}\boldsymbol{\Lambda}^{\frac{1}{k}}\mathbf{U}^*$. The principal root satisfies $\mathbf{R}^{\frac{1}{k}} = \exp(\frac{1}{k}\mathbf{S}) \in \mathsf{SO}(n)$. Moreover, $\mathbf{R}^{\frac{1}{k}} \notin \mathsf{SO}(n) \setminus \mathcal{R}$ if $\mathbf{R} \notin \mathsf{SO}(n) \setminus \mathcal{R}$.

3. Problem statement

From a mathematical perspective it is appealing to strive for generalization. Consider the evolution of a positively oriented n-dimensional orthogonal frame represented by $\mathbf{R} \in \mathsf{SO}(n)$. The dynamics on $\mathsf{SO}(n)$ are given by $\dot{\mathbf{R}} = \boldsymbol{\Omega}\mathbf{R}$. This paper concerns the following system.

System 1. *Consider the system*

$$\dot{\mathbf{R}} = \boldsymbol{\Omega}(\mathbf{R})\mathbf{R} \tag{1}$$

where $\mathbf{R} \in \mathsf{SO}(n)$ and $\boldsymbol{\Omega} : \mathsf{SO}(n) \to \mathsf{so}(n)$. The input is given by $\boldsymbol{\Omega}$, i.e. the system is actuated on a kinematic level.

The kinematic level stabilization problem on $\mathsf{SO}(n)$ concerns the design of an $\boldsymbol{\Omega}$ that stabilizes the identity matrix. Eq. (1) states that $\mathbf{R}$ can be actuated along any direction of $\mathsf{so}(n)$, its tangent space at the identity. Note that $\mathsf{SO}(n)$ is invariant under the kinematics (1), *i.e.* any solution $\mathbf{R}(t; \mathbf{R}_0)$ to (1) for which $\mathbf{R}_0 \in \mathsf{SO}(n)$ remains in $\mathsf{SO}(n)$ for all $t \in [0, \infty)$. This paper concerns a class of almost globally stabilizing feedback laws $\boldsymbol{\Omega}$ that allow (1) to be solved for $\mathbf{R}$ as a function of time, any design parameters, and the initial conditions. It also analyses the stability of said class of control laws and discusses possible applications of these results.

An equilibrium of (1) is said to be almost globally asymptotically stable if it is asymptotically stable and the region of attraction is all of $\mathsf{SO}(n)$ except for a set of measure zero. A set $\mathcal{N} \subset \mathsf{SO}(n)$ has measure zero if for every chart $\phi : \mathcal{S} \to \mathbb{R}^{\frac{1}{2}n(n-1)}$ in some atlas of $\mathsf{SO}(n)$, it holds that $\phi(\mathcal{S} \cap \mathcal{N})$ has Lebesgue measure zero.

Problem 1. For a given almost globally stabilizing feedback law $\boldsymbol{\Omega} : \mathsf{SO}(n) \to \mathsf{so}(n)$, solve System 1 for $\mathbf{R}(t; \mathbf{R}_0)$, *i.e.* for $\mathbf{R}$ as function of the time $t \in [0, \infty)$ and all initial conditions $\mathbf{R}_0 \in \mathsf{SO}(n)$ belonging to the region of attraction of the identity matrix.

Previous work on global level attitude stabilization apply the stable–unstable manifold theorem (Chaturvedi et al., 2011; Lee, 2012; Sanyal et al., 2009) or use Lyapunov function arguments (Mayhew et al., 2011b) to establish the region of attraction of the identity matrix. The stable–unstable manifold theorem (Sastry, 1999) is however ineffective to prove almost global asymptotical stability for systems that are actuated on a kinematic level when the unstable equilibrium manifold corresponds to the uncountable set $\{\mathbf{R} \in \mathsf{SO}(n) \mid \mathbf{R}^\top = \mathbf{R}\} \setminus \{\mathbf{I}\} \subset \mathcal{R}$.

This paper presents a novel approach to establishing almost global asymptotical stability by means of exact solutions to the closed-loop system kinematics. It is possible to establish global existence and uniqueness of the solutions, see Lemma 1 in the Appendix. Statements regarding control performance can hence be based on the properties of the exact solutions. This paper uses the solutions to show that the region of attraction of the identity matrix for the closed-loop systems generated by Algorithms 1–2 is $\mathsf{SO}(n) \setminus \mathcal{R}$. The desired result follows since $\mathcal{R}$ is a set of measure zero in $\mathsf{SO}(n)$.