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A randomized approximation algorithm for the minimal-norm static-output-feedback problem[☆]

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ABSTRACT

A new randomized algorithm is suggested, for extracting static-output-stabilizing-feedbacks, with approximately minimal-norm, for LTI systems. The algorithm has two similar stages, where in the first one the feasibility problem is solved, and in the second one the optimization problem is solved. The formulation is unified for the feasibility and for the optimization problems, as well as for continuous-time or discrete-time systems. The method is demonstrated by applying it to the hard (conjectured to be NP-hard) problem of the minimal-gain static-output-stabilizing-feedback, and to the hard (conjectured to be NP-hard) problem of regional pole-placement via static-output-feedback in non-convex or unconnected regions. A proof of convergence (in probability) that captures the two rounds of the algorithm is given, and complexity analysis is provided, under some mild assumptions.

1. Introduction

The problem of finding necessary and sufficient conditions, for the existence of static-output-stabilizing-feedbacks (SOF), which can be computed in reasonable (i.e. polynomial) time, has a long history (see Syrmos, Abdallah, Dorato, and Grigoradis (1997) for a survey). The problem is known to be NP-hard if structural-constraints or bounds are imposed on the entries of the controllers (see Blondel and Tsitsiklis (1997); Nemirovskii (1993)), but unknown and suspected to be NP-hard, otherwise. Obviously, in real-life, when one searches a minimal-norm SOF, it is always done in a bounded region. Thus, minimal-gain SOF with bounded entries is obviously NP-hard problem. Pole-placement and simultaneous stabilization via SOF are also NP-hard (see Fu (2004) and Blondel and Tsitsiklis (1997), resp.). Thus, practically, one should expect that only approximation or randomized algorithms will be able to cope with these problems.

Many problems can be reduced to the constrained SOF problem. These include the reduction of the minimal-degree dynamic-feedback problem, robust or decentralized stability via static-feedback and PID controllers to this problem (see Blondel and Tsitsiklis (1997); Mesbahi (1999) and Zheng, Wang, and Lee (2002), resp.). A formulation of the reduced-order H_∞ filter problem as constrained SOF problem, is considered in Borges, Calliero, Oliveira, and Peres (2011).

The solution of the SOF problem is important for systems which models structural dynamics, and naturally needs a static feedback that can be built into the structure (e.g. buildings and bridges subject to earthquakes, strong winds and unpredicted dynamic loadings—see Spencer and Sain (1997); Xu and Teng (2002); Yang, Lin, and Jabbari (2003) and Polyak, Khlebnikov, and Shcherbakov (2013), where it is shown that optimal SOF's, although costless, may achieve similar performance as optimal dynamic feedbacks). Note that the long-term memory of dynamic feedbacks is useless in the case of unpredicted dynamic loadings.

The suggested algorithm that will be called the Ray-Shooting (RS) algorithm, does not make use of any heavy tools like Semi-Definite Programming (SDP) or Linear, or Bilinear Matrix Inequalities (LMI's and BMI's resp.). Note that the existence of BMI's solutions is NP-hard problem (see Toker and Özbay (1995)). The least-rank dynamic-output-feedback problem is considered in Mesbahi (1999) using SDP. This can be solved with the RS method, by applying a Binary-Search on the rank and using its formulation as a SOF problem, with much less computational efforts and without the need of finding any feasible initial point. Also, the problem of optimal abscissa via SOF can be solved with the RS method, by applying Binary-Search on the abscissa.

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In Vidyasagar and Blondel (2001) the problem of a common Lyapunov matrix, and the problems of static-stability and simultaneous static stability, are treated, using the probabilistic method (i.e. the "generate and check" strategy). The article deals with the problem of counting the minimal number of samples that guarantee a given probability threshold of success, through the use of the Chernoff bound and bounds on the Vapnik–Chervonenkis dimension (VC-dimension) of the problem, without knowing the specific distribution of successful examples and without using the structure of the given system (i.e. the specific A, B, C matrices). In this article, it is shown that at least $\max\left(\frac{16}{\epsilon^2}\ln\left(\frac{4}{\delta}\right), \frac{32d}{\epsilon^2}\ln\left(\frac{32e}{\epsilon^2}\right)\right)$ samples are needed to guarantee with $1-\delta$ confidence that the empirical probability is ϵ uniformly-close to the exact probability (with $\epsilon, \delta \in (0,1)$), where d is the VC-dimension of the problem, where for the SOF problem it is shown there that $d \le 4p^2\ln\left(4e\left(2p^2+p\right)\right)$ (p being the state-space dimension). Thus, for $p = 10, \epsilon = \delta = 0.01$ at least $1.3538 \cdot 10^{10}$ samples are needed ($d \le 3093$).

One way to overcome this "curse of dimensionality" is presented in Tempo, Calafiore, and Dabbene (2005) and Tempo and Ishii (2007). Concerning the problem of Robust Stability via SOF, it is proved in Tempo et al. (2005) that the empirical performance measure is ϵ_1 probability-close to the exact performance with probability at least $1-\epsilon_2$ and confidence at least $1-\delta$, if one takes $M \ge \frac{\ln(2/\delta)}{\ln(1/(1-\epsilon_2))}$ samples of the controller parameters and $N \ge \frac{\ln\left(\frac{4M}{\delta}\right)}{2\epsilon_1^2}$ samples of the system uncertainty ($\epsilon_1, \epsilon_2, \delta \in (0,1)$). Thus, for $\epsilon_1 = \epsilon_2 = \delta = 0.01$ one needs $M \ge 528$ samples of the controller parameters and $N \ge 61,303$ samples of the system uncertainty, which results in $32,367,984$ evaluations of the performance measure. The RS algorithm seems to practically overcome this obstacle and at least suggests another way to cope with this problem.

The structure of the article is as follows: in Section 2, we set notions, definitions, and give some lemmas. In Section 3, we introduce a lemma which provides the basis for the RS algorithm. We next introduce an approximation algorithm that applies the RS method again, in order to find a minimal-gain SOF. In Section 4, we consider the Alternating-Projections (AP) algorithm introduced in Yang and Orsi (2006), which solves the problem of pole-placement via static-feedback, and we revise this algorithm with some improvement. We also consider the Hide-And-Seek algorithm, introduced in Bélisle (1992), which solves the global optimization problem of continuous functions on compact domains. In Section 5, we compare the results of the RS algorithm with the algorithms: AP, Hide-And-Seek, Mixed LMI/Randomized Method (see Arzelier, Gryazina, Peaucelle, and Polyak (2010)), HIFOO (see Gumussoy, Henrion, Millstone, and Overton (2009)) and HINFSTRUCT (see Apkarian and Noll (2006)). In Section 6, we revise a proof for the convergence in probability of the RS algorithm and discuss its complexity under some reasonable assumptions. In Section 7, we conclude with some remarks concerning the comparison between the algorithms and discuss the limitations of the RS method.

2. Preliminaries

The complex, real and rational fields are denoted by $\mathbb{C}, \mathbb{R}, \mathbb{Q}$, resp. We denote by $\mathbb{D}$ the open unit disk and by $\mathbb{C}_-$ the open left half-plane. For $\Omega \subseteq \mathbb{C}$ we denote by $\overline{\Omega}$ the set $\mathbb{C} \setminus \Omega$. For $z \in \mathbb{C}$ we denote by $\Re(z), \Im(z)$ its real and imaginary parts, resp. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For a set $S \subset \mathbb{C}^{p\times 1}$, we denote by $Span(S)$ the linear span of S. For a square matrix Z, we denote by $\sigma(Z)$ the spectrum of Z. For a $\mathbb{C}^{p\times q}$ matrix Z, we denote by Z^* its conjugate transpose, and by $Z_{i,j}$ its (i,j)'th element or block element. A square matrix Z is said to be non-negative (denoted as $Z \ge 0$) if $Z^* = Z$ and $v^*Zv \ge 0$ for any v. A non-negative matrix Z is said to be strictly non-negative (denoted as $Z > 0$) if $v^*Zv > 0$ for any $v \ne 0$. For two square matrices Z, W we write $Z \ge W$ ($Z > W$) if $Z - W \ge 0$ (resp., if $Z - W > 0$). For a matrix $Z \in \mathbb{C}^{p\times q}$, we denote by Z^+ the Moore–Penrose Pseudo-inverse (see Karlheinz (1994); Piziak and Odell (2007)). By L_Z, R_Z we denote the orthogonal projections $I_q - Z^+Z$ and $I_p - ZZ^+$ resp., where I_t denotes the identity $t\times t$ matrix. Note that Z^+Z and ZZ^+ (as well as L_Z and R_Z) are self-adjoint and unitarily diagonalizable with $\{0,1\}$ eigenvalues. In what follows, we assume that the given systems have the form:

$$\begin{cases} \Sigma(x) = Ax + Bu \\ \quad y = Cx \end{cases} \tag{1}$$

where $\Sigma(x(t)) = \dot{x}(t)$ for continuous-time systems and $\Sigma(x_k) = x_{k+1}$ for discrete-time systems, where $A \in \mathbb{C}^{p\times p}, B \in \mathbb{C}^{p\times q}, C \in \mathbb{C}^{r\times p}$. A matrix Z in the discrete-time context, is said to be α-stable, for $0 < \alpha < 1$, if any eigenvalue $\lambda \in \sigma(Z)$ satisfies $|\lambda| < \alpha$. A pair (A, B) is said to be α-stabilizable, if there exists K for which $E = A - BK$ is α-stable or, equivalently, if there exists X for which $E = A - BB^+X$ is α-stable. Note that (A, B) is α-stabilizable if and only if $rank\begin{bmatrix}\lambda I - A & B\end{bmatrix} = p$, for any λ satisfying $|\lambda| \ge \alpha$. Equivalently, (A, B) is α-stabilizable if and only if there exist X and $P, R > 0$ for which

$$P - \left(\frac{E^*}{\alpha}\right) P \left(\frac{E}{\alpha}\right) = R. \tag{2}$$

Similarly, a square matrix Z in the continuous-time context is said to be α-stable, for $\alpha > 0$, if any eigenvalue $\lambda \in \sigma(Z)$ satisfies $\Re(\lambda) < -\alpha$. Thus, (A, B) is α-stabilizable if and only if there exist X and $P, R > 0$ for which

$$(E + \alpha)^* P + P(E + \alpha) = -R, \tag{3}$$

where $E = A - BB^+X$. The last two examples of restricted stability region, can be generalized as follows. Let Ω_0 be $\mathbb{D}$ or $\mathbb{C}_-$. Let $\Omega \subset \Omega_0$ be any simply connected region with smooth boundary. Let $\varphi : \Omega_0 \to \Omega$ be a conformal, one-to-one mapping, in which its existence is guarantied by the Riemann Mapping Theorem (see W. Rudin Rudin (1987), Theorem 14.8, p. 283). Let $\varphi^{-1} : \Omega \to \Omega_0$ denote its inverse and assume that the substitution $\varphi^{-1}(Z)$ is meaningful for matrix Z in some open set and that $\varphi^{-1}(Z)v = \varphi^{-1}(\lambda)v$ for any eigenvalue λ and related eigenvector v of Z. For example, if $\varphi^{-1}(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z)$ are polynomials then, a substitution of a square matrix Z into $\varphi^{-1}(z)$ will be calculated as: $p(Z)q(Z)^{-1} = q(Z)^{-1}p(Z)$, where we assume that $q(Z)$ is invertible. Let λ be an eigenvalue of Z with eigenvector v, then $p(Z)v = p(\lambda)v$ and $q(Z)^{-1}v = \frac{1}{q(\lambda)}v$ implying $\varphi^{-1}(Z)v = \varphi^{-1}(\lambda)v$. Now, a square matrix Z is said to be Ω-stable, if $\sigma(Z) \subset \Omega$, and we write $\mathcal{S}_\Omega^{p\times p}$ for the set of all Ω-stable $p\times p$ matrices. A pair (A, B) is said to be Ω-stabilizable, if there exists X for which $E = A - BB^+X$ is Ω-stable. One can prove that (A, B) is Ω-stabilizable if, and only if $rank\begin{bmatrix}\lambda I - A & B\end{bmatrix} = p$ for any $\lambda \in \overline{\Omega}$. The last is equivalent, in the discrete-time context, to the existence of X and $P, R > 0$, such that

$$P - \varphi^{-1}(E)^* P \varphi^{-1}(E) = R. \tag{4}$$

In the continuous-time context, (A, B) is Ω-stabilizable if, and only if there exist X and $P, R > 0$ for which

$$\varphi^{-1}(E)^* P + P\varphi^{-1}(E) = -R. \tag{5}$$

Related inequalities for robust pole-placement in LMI regions and H_∞ design with pole-placement in LMI regions, are considered in Chilali and Gahinet (1996) and Chilali, Gahinet, and Apkarian