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Fractional order control of the two-dimensional wave equation ☆

Lea Sirota [1], Yoram Halevi

Faculty of Mechanical Engineering, Technion I.I.T, Haifa, Israel

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ABSTRACT

Control of systems governed by the two-dimensional linear wave equation in finite spatial domain is considered and presented through vibrating rectangular membranes. The membranes are modeled by modal decomposition in one spatial axis and infinite dimensional transfer functions in the other. The transfer functions are built of fractional order exponents, regarded as non-pure delays, which are shown to represent traveling waves whose shape changes during motion. The membranes are controlled in closed loop to achieve position profile tracking and attenuation of disturbances. The actuation is along two opposite boundaries, which controls the entire wave motion between them. The control algorithm stops the wave propagation in the control axis by creating active non-reflecting boundaries. In addition, it compensates the remaining non-pure delay by extending the classical dead time compensation principle. As a result, despite the infinite dimension of the system and its fractional order transfer functions, the closed loop transfer function is given by a rational first order lag with a pure time delay. The resulting controllers are also of fractional order and their implementation is obtained by dedicated approximations. The system stability with the approximated controllers is investigated formally using robustness tools. The control algorithm is demonstrated by means of several examples.

1. Introduction

The linear wave equation is encountered in diverse fields of engineering such as mechanics, acoustics, and electromagnetics. In mechanics, the wave equation describes the vibrations of flexible structures that have no resistance to bending, (Kreyszig, 2010). In the case of a single spatial dimension (1D) these are vibrations of strings or rods, whereas for two dimensions (2D) these are vibrations of membranes. Membranes are widely employed in space applications due to their light weight and low packing volume, (Jenkins, 2001; Ruggiero & Inman, 2006). However, flexibility has adverse effects as well. Tracking maneuvers or the action of disturbances will excite the flexible modes, resulting in an undesired vibration. While passive rigidization would simply result in heavier structures, the alternative is active rigidization via control algorithms.

Since the wave equation is a partial differential equation (PDE), the use of standard control methods is not straightforward. Several methods to control systems governed by PDE's exist (see, e.g. the review by Padhi and Ali (2009)). One common practice is first to approximate the continuous medium by spatially discrete models, which can be achieved either by modal truncation, by the Finite Element (FE) method or even by experimentally obtained frequency response, and then to control it by state space techniques. This is the approach, for example, in Balas (1978) and Hu (2008) for the control of 1D flexible structures and in Kukathasan and Pellegrino (2002) and Sakamoto, Park, and Miyazaki (2006) for the control of membranes. However, the use of approximated models carries some disadvantages. First, for high accuracy the approximations should have very high order, which practically prevents the use of common systematic control methods. Secondly, the system physical characteristics, such as traveling waves and the associated delays are lost in the approximation process. In methods that are not based on approximations the controller is incorporated directly in the PDE. Such methods are used in Guo and Jin (2013) and Krstic, Guo, Balogh, and Smyshlyaev (2008) for stabilization of an unstable wave equation.

A different approach is modeling the continuous systems by infinite dimensional (irrational) transfer functions (TF), which arise from applying the Laplace transform directly to the PDE (see a tutorial by Curtain and Morris (2009)). This approach is adopted

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E-mail addresses: lsirota@technion.ac.il (L. Sirota), yoramh@technion.ac.il (Y. Halevi).

[1] Tel.: +972 4 8292095.

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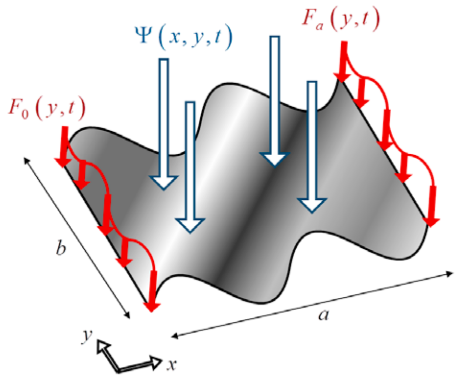

Fig. 1. A rectangular membrane (illustration).

e.g. in Alli and Singh (2000) to control the 1D wave equation by root locus and optimization methods, or in Saito and Katsura (2013) to control a multi-mass resonant system by wave absorption strategy. In a series of publications (Halevi, 2005; Halevi & Peled, 2010; Sirota & Halevi, 2010, 2012), the infinite dimensional TFs were used to model flexible rods in torsion (1D wave equation) with a general linear set of boundary conditions (BC). The TF model consisted of exponents that are linear in s and of low order rational terms. These pure time delay exponents exhibit the traveling waves phenomena in the structure, whereas the rational parts represent the reflection of waves from the boundaries. The model was then used to design a dedicated control algorithm (with boundary actuation) that stops the wave reflections, achieving absolute vibration suppression (AVS) in the system.

The goal of this work is to extend the infinite dimensional TF modeling and the consequent wave based control approach to systems governed by the 2D wave equation, such as the membrane shown in Fig. 1. However, the additional spatial dimension complicates matters, since unlike the 1D wave equation, application of Laplace transform in the 2D case still yields a PDE. Our approach is therefore to represent the system by a combination of modal decomposition in one spatial direction (the lateral axis y) and of infinite dimensional TFs in the other (the longitudinal axis x). These TFs consist of fractional order exponents, which represent non-pure time delays and describe the special manner of wave propagation in the system. We exploit this model to track the structure position in closed loop using actuation along the boundaries $x = 0$ and $x = a$. More specifically, we wish to track the structure position at the x center-line along a desired y profile. For each controlled y mode our strategy is first to eliminate the wave reflections in the x axis and then to compensate for the remaining non-pure delay. This requires the extension of the AVS method and of the standard dead time compensation algorithm, (Smith, 1957), to TFs with fractional order exponents.

2. System model and open loop response

2.1. Problem statement and solution in Laplace domain

As a representative example we consider a taut rectangular membrane of length a and width b, as illustrated in Fig. 1. Assuming small deflections and negligible internal damping, the membrane is governed by the 2D wave equation,

$$\frac{1}{c^2}\frac{d^2}{dt^2}W(x,y,t) - \frac{d^2}{dx^2}W(x,y,t) - \frac{d^2}{dy^2}W(x,y,t) = \frac{1}{T}\Psi(x,y,t), \tag{1}$$

where $W(x,y,t)$ is the vertical deflection, $c = \sqrt{T/\rho}$ is the wave propagation velocity, ρ is the areal density and T is the tension in the x–y plane. The membrane is subjected to a distributed force $\Psi(x,y,t)$ (disturbance) and to the boundary forces $F_0(y,t)$ and $F_a(y,t)$ along $x = 0$ and $x = a$ (control action), respectively. The boundaries $y = 0$ and $y = b$ are free and the BC are therefore given by

$$-T\frac{d}{dx}W(0,y,t) = F_0(y,t), \tag{2a}$$

$$T\frac{d}{dx}W(a,y,t) = F_a(y,t), \tag{2b}$$

$$\frac{d}{dy}W(x,0,t) = \frac{d}{dy}W(x,b,t) = 0. \tag{2c}$$

We are interested in a traveling wave form of the response, hence we attend the problem via Laplace domain. Laplace transform with respect to time of (1) gives

$$\left(\frac{s}{c}\right)^2 W(x,y;s) - \frac{d^2}{dx^2}W(x,y;s) - \frac{d^2}{dy^2}W(x,y;s) = \frac{1}{T}\Psi(x,y;s), \tag{3}$$

where $W(x,y;s)$ is the transform of $W(x,y,t)$, etc. Similarly, we apply the Laplace transform to the BC. As was stated in the introduction, we model the system by a combination of modal decomposition in one spatial direction and infinite dimensional TFs in the other. Let the modal direction be the lateral, we suggest the solution of (3) in the form

$$W(x,y;s) = \sum_{n=0}^{\infty} w_n(x;s)\varphi_n(y). \tag{4}$$

Substituting (4) in (3) and considering its homogeneous part (i.e. $\Psi(x,y;s) = 0$), the nth solution of the lateral component becomes

$$\varphi_n(y) = A_n e^{iq_n y} + B_n e^{-iq_n y} \tag{5}$$

for some constant A_n, B_n and q_n. Applying to (5) the homogeneous BC (2c), we obtain

$$A_n = B_n, \qquad q_n = \frac{n\pi}{b}, \qquad \varphi_n(y) = \cos(q_n y), \tag{6}$$

where q_n and $\varphi_n(y)$ are the lateral eigenvalues and eigenfunctions. A_n was set arbitrarily to unity as it will be merged with the coefficients of $w_n(x;s)$. Expanding the distributed force by $\varphi_n(y)$, gives

$$\Psi(x,y;s) = \sum_{n=0}^{\infty} \psi_n(x;s)\varphi_n(y), \tag{7}$$

which is a cosine Fourier series with the coefficients $\psi_n(x;s)$. Substituting (7) in (3) and using the fact that $d^2/dy^2\varphi_n(y) = -q_n^2\varphi_n(y)$, $\varphi_n(y)$ can be eliminated. The nth equation of the longitudinal part then becomes

$$\frac{d^2}{dx^2}w_n(x;s) - \left(\frac{\alpha_n(s)}{c}\right)^2 w_n(x;s) + \frac{1}{T}\psi_n(x;s) = 0, \tag{8}$$

where

$$\alpha_n(s) = \sqrt{s^2 + c^2 q_n^2}. \tag{9}$$

The solution of ODE (8), which is the nth longitudinal open loop response component, is given by

$$w_n(x;s) = C_{1n}(s)e^{\frac{x}{c}\alpha_n(s)} + C_{2n}(s)e^{-\frac{x}{c}\alpha_n(s)} - \frac{c}{2T\alpha_n(s)}\int_0^x \left(e^{\frac{x-\xi}{c}\alpha_n(s)} - e^{-\frac{x-\xi}{c}\alpha_n(s)}\right) \times \psi_n(\xi;s)d\xi. \tag{10}$$