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Clustered model reduction of positive directed networks[☆]

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ABSTRACT

This paper proposes a clustered model reduction method for semistable positive linear systems evolving over directed networks. In this method, we construct a set of clusters, i.e., disjoint sets of state variables, based on a notion of cluster reducibility, defined as the uncontrollability of local states. By aggregating the reducible clusters with aggregation coefficients associated with the Frobenius eigenvector, we obtain an approximate model that preserves not only a network structure among clusters, but also several fundamental properties, such as semistability, positivity, and steady state characteristics. Furthermore, it is found that the cluster reducibility can be characterized for semistable systems based on a projected controllability Gramian that leads to an a priori $\mathcal{H}_2$-error bound of the state discrepancy caused by aggregation. The efficiency of the proposed method is demonstrated through an illustrative example of enzyme-catalyzed reaction systems described by a chemical master equation. This captures the time evolution of chemical reaction systems in terms of a set of ordinary differential equations.

1. Introduction

Many dynamical systems of interest to control community are inherently constructed from subsystem interconnections. Examples of such interconnected systems include power grids, transportation networks and so forth; see Boccaletti, Latora, Moreno, Chavez, and Hwang (2006) for an overview. Since the network structure of such systems is often complex and large-scale, it is crucial to develop an approximation method that enables us to reduce their complexity (dimension). In addition, it is more desirable to preserve some particular properties of these systems, such as a network structure, stability, and positivity, throughout the approximation. This kind of structure-preserving model reduction has the potential to significantly simplify to analyze large-scale systems while capturing their essential properties of interest.

A number of model reduction methods can be found in the literature (Antoulas, 2005). For instance, model reduction methods inspired by principal component analysis, such as the balanced truncation (Enns, 1984) and the Hankel norm approximation (Lin & Chiu, 1990), are well known. A major advantage of these methods is the availability of an error bound in terms of the $\mathcal{H}_\infty$-norm or Hankel norm. Furthermore, the class of moment matching methods, including the Krylov subspace methods, is also well known (Gugercin & Willcox, 2008). This class of methods aims to suppress discrepancies in the system behavior for specific input signals, and has the advantage of a computationally efficient implementation. However, unlike the former class of methods, a priori error bounds have not yet been derived. For these existing model reduction, a systematic procedure is provided. However, they have a drawback in terms of their application to network system: the network structure of systems, i.e., the interconnection topology among state variables or subsystems, is

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destroyed through the approximation. This is because each state of the resultant approximants is constructed by a linear combination of all the original states. Therefore, to practically approximate a network system, it is crucial to develop a model reduction method that explicitly preserves the network structure of the system.

One approach to network structure-preserving model reduction can be an extension of structure-preserving model reduction methods found in the literature. For example, van der Schaft and Polyuga (2009) address a model reduction problem that considers the preservation of the second-order structure. However, this problem is not formulated on the premise of network structure preservation. In contrast to the existing approach, a clustered model reduction method has been developed for stable systems evolving over undirected, or bidirectional networks (Ishizaki, Kashima, Imura, & Aihara, 2014). In this method, by focusing on the symmetry of system matrices, we have introduced a system transformation, called positive tridiagonalization, to characterize the cluster reducibility, defined as the uncontrollability of disjoint subsets, or clusters, of state variables. The aggregation of reducible clusters yields an approximate model that preserves the network structure among clusters and the stability of systems, and provides an error bound in terms of the $\mathcal{H}_\infty$-norm. However, the applicability of this clustered model reduction is rather restricted because both stability preservation and reducibility characterization are heavily reliant on the symmetry of the system matrices. From a practical point of view, it is crucial to improve the applicability of our clustered model reduction framework.

One major difficulty confronted by network structure-preserving model reduction involves preserving the stability of the original system in its approximants. To enable the systematic development of clustered model reduction, it is important to clarify the class of systems to which it can reasonably be applied. In Farina and Rinaldi (2000), it has been found that stability analyses can be tractably performed for a class of systems admitting a positive property, called (internally) positive systems. More specifically, the stability of positive systems can be characterized by an eigenpair, called the Frobenius eigenvalue and eigenvector. In fact, clustered model reduction has good compatibility with the approximation of positive systems because, as long as we make the aggregation coefficients non-negative, the positivity property of systems can be preserved in its approximants. In this paper, we use this compatibility to show that the semistability of positive systems can be preserved by a selection of aggregation coefficients specified by the Frobenius eigenvector. Moreover, we derive an alternative characterization of cluster reducibility based on a projected controllability Gramian. Owing to this development, we can apply clustered model reduction to semistable positive systems, called positive directed networks, involving compartmental systems, and Markovian processes (Farina & Rinaldi, 2000).

To demonstrate the improved applicability, we provide an illustrative example of a chemical master equation (CME) compatible with enzyme-catalyzed reaction systems. It is known that CMEs belong to a class of Markovian processes (Higham, 2008; Munsky & Khammash, 2008), which can be regarded as a semistable positive directed network. Since the dimension of CMEs tends to be large, they are not necessarily analytically or numerically tractable. To overcome this difficulty, the proposed clustered model reduction method produces an aggregated model that preserves several fundamental properties as Markovian processes. A preliminary version of this paper was published in Ishizaki et al. (2012). In comparison with it, this paper provides detailed proofs and explanations for our theoretical results.

The remainder of this paper is structured as follows: In Section 2, we first formulate a clustered model reduction problem for positive directed networks. In Section 3, we characterize the cluster reducibility using a projected controllability Gramian, and develop a clustered model reduction method. Section 4 demonstrates

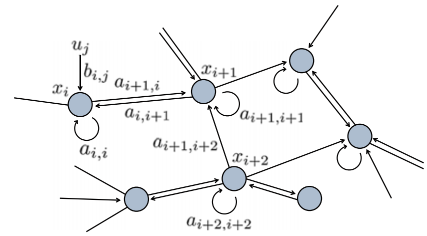

Fig. 1. Depiction of positive directed networks.

the efficiency of the proposed method through an illustrative example of CMEs. Finally, concluding remarks are provided in Section 5.

Notation $\mathbb{R}$: the set of real numbers, $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$): the set of positive (non-negative) real numbers, $\mathbb{N}$: the set of non-negative integers, I_n: the n-dimensional identity matrix, $|\mathcal{I}|$: the cardinality of a set $\mathcal{I}$, $\mathrm{im}(M)$: the image of a matrix M, $\|M\|_{\mathrm{F}}$: the Frobenius norm of a matrix M, $\mathrm{diag}(v)$: the diagonal matrix having a vector v on its diagonal, $\mathrm{Diag}(M_1, \ldots, M_n)$: the block diagonal matrix having matrices $M_1, \ldots, M_n$ on its block diagonal.

For $\mathcal{I} \subseteq \{1, \ldots, n\}$, let $e_{\mathcal{I}}^n \in \mathbb{R}^{n \times |\mathcal{I}|}$ denote the matrix composed of the column vectors of I_n compatible with $\mathcal{I}$. A square matrix M (respectively, a transfer matrix G) is said to be *semistable* if all eigenvalues of M (poles of G) are in the closed left-half plane, and all eigenvalues (poles) with zero real value are simple roots. A square matrix M is said to be *reducible* if it can be placed into block upper-triangular form by simultaneous row and column permutations. Conversely, M is said to be *irreducible* if it is not reducible. Furthermore, M is said to be *Metzler* if the off-diagonal entries of M are all non-negative. The positive (negative) semidefiniteness of $M = M^{\mathsf{T}} \in \mathbb{R}^{n \times n}$ is denoted by $M \succeq O_n$ ($M \preceq O_n$). Its positive (negative) definiteness is denoted similarly. The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G and the $\mathcal{H}_2$-norm of a stable strictly proper transfer matrix G are denoted by $\|G\|_{\mathcal{H}_\infty}$ and $\|G\|_{\mathcal{H}_2}$.

2. Problem formulation

2.1. Preliminaries

In this paper, we deal with a class of positive linear systems evolving over directed networks. We denote a set of irreducible Metzler matrices by

$$\mathbb{M}_n := \{M \in \mathbb{R}^{n \times n} : \text{irreducible, Metzler}\}. \tag{1}$$

In this notation, we define the following class of positive systems:

Definition 1. A linear system

$$\Sigma : \dot{x} = Ax + Bu \tag{2}$$

is said to be a *positive directed network* if $A \in \mathbb{M}_n$ and $B \in \mathbb{R}_{\geq 0}^{n \times m}$.

This class of systems includes spatially-discrete reaction–diffusion systems, electrical circuit networks, continuous-time Markovian processes, and so forth. Their state trajectory does not escape from the non-negative orthant $\mathbb{R}_{\geq 0}^n$ under non-negative input signals and initial conditions. Such systems having the non-negative property often appear in science and engineering (Farina & Rinaldi, 2000). With the notation of $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$, Fig. 1 depicts the interconnection topology (network structure) of positive directed networks. Note that the irreducibility of $A \in \mathbb{M}_n$ assumed in (1) coincides with the strong connectivity of networks, which can be relaxed under a suitable situation; see Section 3.4 for details.