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Brief paper

Hysteresis loop of the LuGre model ☆

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ABSTRACT

The LuGre friction model is used in the current literature to describe the friction phenomenon for mechanical systems. In this paper, we focus on the hysteresis behaviour of the model. More precisely, we describe analytically the hysteresis loop of the model through the concepts of consistency and strong consistency. The description is illustrated by numerical simulations.

1. Introduction

Friction is a nonlinear phenomenon that originates from the contact of two bodies. It has two types of characteristics, static and dynamic. The static characteristics of friction include the stiction friction, the kinetic force (the Coulomb force), the viscous force, and the Stribeck effect which are functions of steady state velocity. The static friction models give the friction force as a function of velocity and only describe the steady-state behaviour between velocity and friction force. Static friction models are discontinuous at zero velocity with a dependence on the sign of velocity (Armstrong-Hélouvry, Dupont, & Canudas De Wit, 1994).

This discontinuity does not reflect accurately the real friction behaviour and may cause errors in numerical simulations, or even instability in the algorithms designed to compensate friction (Armstrong-Hélouvry et al., 1994).

Dynamic friction models capture properties that cannot be captured by typical static friction models; for instance, presliding displacement related to the elastic and plastic deformations of asperities, frictional lag, that is the delay in the change of friction force as a function of a change of velocity, and stick–slip motion. These models do not present a discontinuity at zero velocity which makes them more suitable for numerical simulations and friction compensation (Armstrong-Hélouvry et al., 1994).

Dahl friction model is a dynamic model whose steady-state is the Coulomb friction (Dahl, 1976). The main contribution of the model is that it takes into account the existence of hysteresis between the presliding friction force input and the displacement output that is observed experimentally (Armstrong-Hélouvry et al., 1994). However, Dahl model does not capture the Stribeck effect. An improvement of this model is implemented in the LuGre model (Canudas de Wit, Olsson, Åström, & Lischinsky, 2000) which captures some essential properties of friction such as hysteresis and Stribeck effect and thus can describe stick–slip motion (Åström & Canudas-de-Wit, 2008). Therefore, it has been used to describe the friction phenomenon for mechanical systems (Åström & Canudas-de-Wit, 2008; Padthe et al., 2008). Necessary and sufficient conditions for the dissipativity of the LuGre model are given in Barahanov and Ortega (2000). Also, the model has been used for friction compensation (Freidovich, Robertsson, Shiriaev, & Johansson, 2010; San, Ren, Ge, Lee, & Liu, 2011; Swevers, Al-Bender, Ganseman, & Projogo, 2000).

In this paper, we focus on the hysteresis behaviour of the LuGre model. More precisely, we investigate the analytical expression of the hysteresis loop of the model through the concepts of consistency and strong consistency (Ikhouane, 2013). These concepts are particularly useful when dealing with *rate-dependent* hysteresis as is the case of the LuGre model. The reader is referred to Ikhouane (2013) for a more detailed explanation and motivation of the concepts of consistency and strong consistency.

The paper is organized as follows. Section 2 presents the needed background from Ikhouane (2013). The problem statement

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is formulated in Section 3. The main results of this paper are presented in Section 4. These results are commented upon is Section 5, and a simulation example is provided in Section 6. The conclusion is given in Section 7.

2. Background results

This section summarizes the results obtained in Ikhouane (2013).

2.1. Class of inputs

The Lebesgue measure on $\mathbb{R}$ is denoted μ. For a measurable function $p : I \subset \mathbb{R}_+ \to \mathbb{R}^m$, $\|p\|_{\infty,I}$ denotes the essential supremum of $|p|$ on I where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. When $I = \mathbb{R}_+$, it is denoted $\|p\|_\infty$.

Consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u : \mathbb{R}_+ \to \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative $\dot{u}$ is defined a.e., and we have $\|u\|_\infty < \infty$, $\|\dot{u}\|_\infty < \infty$.

For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \to \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)|\,d\tau \in \mathbb{R}_+$. The function ρ_u is well defined, nondecreasing and absolutely continuous. Observe that ρ_u may not be invertible. Let I_u be the range of ρ_u.

Lemma 1 (Ikhouane, 2013). *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant so that I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$ that satisfies $\psi_u \circ \rho_u = u$. The function ψ_u satisfies $\|\dot{\psi}_u\|_{\infty,I_u} = 1$ and $\mu\left[\left\{\varrho \in I_u / \dot{\psi}_u(\varrho) \text{ is not defined or } |\dot{\psi}_u(\varrho)| \neq 1\right\}\right] = 0$.*

Lemma 2 (Ikhouane, 2013). *Define $s_\gamma(t) = t/\gamma, \forall \gamma > 0, t \geq 0$. Then $\forall \gamma > 0$, $I_{u \circ s_\gamma} = I_u$ and $\psi_{u \circ s_\gamma} = \psi_u$.*

2.2. Class of operators

Let Ξ be a set of initial conditions and consider the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. The operator $\mathcal{H}$ is said to be causal if (Visintin, 1994, p. 60): $\forall \left(u_1, \xi^0\right), \left(u_2, \xi^0\right) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi, \forall \tau > 0$ if $u_1 = u_2$ on $[0, \tau]$ then $\mathcal{H}\left(u_1, \xi^0\right) = \mathcal{H}\left(u_2, \xi^0\right)$ on $[0, \tau]$.

Assumption 3 (Ikhouane, 2013). Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ and $y = \mathcal{H}\left(u, \xi^0\right) \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$; if $\exists \theta \in \mathbb{R}_+$ such that u is constant on $[\theta, \infty)$, then y is constant on $[\theta, \infty)$.

Lemma 4 (Ikhouane, 2013). *Assume that $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ is causal and satisfies Assumption 3. Then, $\exists! \varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ that satisfies $\varphi_u \circ \rho_u = y$. Moreover $\|\varphi_u\|_{\infty,I_u} \leq \|y\|_\infty$. If y is continuous on $\mathbb{R}_+$, then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty,I_u} = \|y\|_\infty$.*

2.3. Definition of consistency and strong consistency

Definition 5 (Ikhouane, 2013). Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. The operator $\mathcal{H}$ is said to be consistent with respect to (u, ξ^0) if the sequence of functions $\{\varphi_{u \circ s_\gamma}\}_{\gamma > 0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$ as $\gamma \to \infty$. Denote $L^\infty(I_u, \mathbb{R}^m) \ni \varphi_u^\star = \lim_{\gamma \to \infty} \varphi_{u \circ s_\gamma}$.

Observe that, in Definition 5 of consistency, the input u needs not be periodic. Now, to characterize the hysteresis loop of the operator $\mathcal{H}$ we introduce the concept of strong consistency.

Definition 6 (Ikhouane, 2013). Let $T > 0$. A T-periodic function $w : \mathbb{R}_+ \to \mathbb{R}$ is said to be wave periodic if there exists some $T^+ \in (0, T)$ such that

- The function w is continuous on $\mathbb{R}_+$.
- The function w is continuously differentiable on $\left(0, T^+\right)$ and on $\left(T^+, T\right)$.
- The function w is increasing on $\left(0, T^+\right)$ and is decreasing on $\left(T^+, T\right)$.

Lemma 7 (Ikhouane, 2013). *If the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is non-constant and T-periodic, then $I_u = \mathbb{R}_+$ and $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is $\rho_u(T)$-periodic. Furthermore, if $n = 1$ and u is wave periodic, then a more precise result can be stated. The function ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in \left(0, \rho_u\left(T^+\right)\right)$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in \left(\rho_u\left(T^+\right), \rho_u(T)\right)$.*

$\forall k \in \mathbb{N}$, let $\varphi_{u,k}^\star \in L^\infty\left([0, \rho_u(T)], \mathbb{R}^m\right)$ be defined as $\varphi_{u,k}^\star(\varrho) = \varphi_u^\star(\rho_u(T)k + \varrho), \forall \varrho \in [0, \rho_u(T)]$.

Definition 8 (Ikhouane, 2013). Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \to L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 3. The operator $\mathcal{H}$ is said to be strongly consistent with respect to (u, ξ^0) if it is consistent with respect to (u, ξ^0), and the sequence of functions $\varphi_{u,k}^\star$ converges in $L^\infty\left([0, \rho_u(T)], \mathbb{R}^m\right)$ as $k \to \infty$. Define $L^\infty\left([0, \rho_u(T)], \mathbb{R}^m\right) \ni \varphi_u^\circ = \lim_{k \to \infty} \varphi_{u,k}^\star$.

If the operator $\mathcal{H}$ is strongly consistent with respect to (u, ξ^0), then the graph $\left\{\left(\psi_u(\varrho), \varphi_u^\circ(\varrho)\right), \varrho \in [0, \rho_u(T)]\right\}$ represents the so-called hysteresis loop.

3. Problem statement

The LuGre model is given by Åström and Canudas-de-Wit (2008):

$$\dot{x}(t) = -\sigma_0 \frac{|\dot{u}(t)|}{g(\dot{u}(t))} x(t) + \dot{u}(t), \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$F(t) = \sigma_0 x(t) + \sigma_1 \dot{x}(t) + f(\dot{u}(t)), \tag{3}$$

where $t \geq 0$ denotes time; the parameters $\sigma_0 > 0$ and $\sigma_1 > 0$ are respectively the stiffness and the microscopic damping friction coefficients; the function $g \in C^0(\mathbb{R}, \mathbb{R})$ represents the macro-damping friction with $g(\vartheta) > 0, \forall \vartheta \in \mathbb{R}$; $x(t) \in \mathbb{R}$ is the average deflection of the bristles; $x_0 \in \mathbb{R}$ is the initial state; $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ is the relative displacement and is the input of the system; $F(t)$ is the friction force and is the output of the system; and $f \in C^0(\mathbb{R}, \mathbb{R})$.

In Eq. (1), the function $g(\dot{u})$ is measurable, thus, the differential equation (1) can be seen as a linear time-varying system that satisfies all assumptions of Filippov (1988, Theorem 3). This implies that a unique absolutely continuous solution of (1) exists on $\mathbb{R}_+$.

On the other hand, define $M_u = \sup_{\alpha \in [-\|\dot{u}\|_\infty, \|\dot{u}\|_\infty]} g(\alpha)$. Then $0 < M_u < \infty$ since g is continuous and $g(\vartheta) > 0, \forall \vartheta \in \mathbb{R}$. We also have $0 < g(\dot{u}(t)) \leq M_u$ for almost all $t \geq 0$. Thus it follows from Åström and Canudas-de-Wit (2008) that $|x(t)| \leq \frac{M_u}{\sigma_0}, \forall t \geq 0$ if $|x_0| \leq \frac{M_u}{\sigma_0}$. If $|x_0| > \frac{M_u}{\sigma_0}$ then $|x(t)| \leq |x_0|, \forall t \geq 0$. Thus, $\forall x_0 \in \mathbb{R}$ we have $x \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$.

Now, in Eqs. (1)–(3), consider the operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}(u, x_0) = F$. Then it can be shown that $\mathcal{H}$ is causal and satisfies Assumption 3. This implies that the concepts introduced in Section 2 apply.