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Brief paper

New sufficient conditions for observer-based control of fractional-order uncertain systems ☆

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ABSTRACT

New simple linear matrix inequalities are proposed to ensure the stability of a class of uncertain fractional-order linear systems by means of a fractional-order deterministic observer. It is shown that the conditions of existence of an observer-based feedback can be split into a set of linear matrix inequalities that are numerically tractable. The presented results show that it is possible to decouple the conditions containing the bilinear variables into separate conditions without imposing equality constraints or considering an iterative search of the controller and the observer gains. Simulations results are given to approve the efficiency and the straightforwardness of the proposed design.

1. Introduction

Fractional-order calculus has a long history and its serves as a modern powerful tool in analyzing various physical phenomena. The interest in understanding systems governed by fractional-order differential equations has grown up during the last decades and many associated results have been appeared, see e.g., Farges, Moze, and Sabatier (2010), Manabe (1960), Matignon (1996), Oustaloup (1983, 1995) and Trigeassou, Maamri, Sabatier, and Oustaloup (2011). It was found that diffusion processes and biological systems can be modeled in terms of fractional-order differential equations, see e.g., Oustaloup (2014) and Sabatier, Agrawal, and Machado (2007). Additionally, the use of fractional-order derivatives and integrals in feedback design has been successful to a large extent in improving the robustness of the closed-loop systems.

Nevertheless, fractional differential equations have not yet received the same attention as ordinary differential equations in the investigation of their stability, simulation, and analysis. Owing to the lack of effective analytic methods for the time-domain analysis and simulation of linear feedback fractional-order systems, a numerical simulation scheme is developed in Hwang, Leu, and Tsay (2002). Exact calculation of fractional-order derivatives of some particular polynomial signals is discussed in Samadi, Ahmad, and Swamy (2004). Stability of dynamical systems, represented by fractional order derivatives, has been investigated using the Routh–Hurwitz criteria, the pole placement method, and Lyapunov strategies. For linear fractional-order systems, it was found that the stability is equivalent to the repartition of the system poles in a restricted area of the complex plane. Based on this key formulation of stability and the use of convex-optimization algorithms, stated as linear-matrix-inequality conditions, numerous sufficient conditions have been proposed to ensure robust stability of some classes of fractional-order type systems, see e.g., Ahn and Chen (2008), Farges et al. (2010) and Sabatier, Moze, and Farges (2010). A considerable interest has been also devoted to stability and stabilizability of special classes of fractional-order systems, see e.g., Li, Chen, and Podlubny (2010) and Wen, Wu, and Lu (2008). A new Lyapunov stability analysis of fractional differential equations is discussed in Trigeassou et al. (2011). The problem of pseudo-state feedback stabilization of fractional-order systems using LMI setting was addressed in Farges et al. (2010).

In the recent paper (Lan, Huang, & Zhou, 2012), the authors have presented a numerical scheme for stabilization of uncertain commensurate ($1 < \alpha < 2$) fractional-order systems by means of dynamic output feedback. In Lan and Zhou (2013), observer-based control of a class of uncertain fractional-order systems $0 < \alpha < 1$ is studied using convex-optimization tools. Other

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recent works on stabilization of triangular fractional-order systems can be traced in Zhang, Liu, Feng, and Wang (2013) and references therein. In this paper, we devote our attention to the control of commensurate fractional-order-pseudo-state systems subject to bounded uncertainties and partially-state measurements where the non integer differentiation order is between zero and one. Additionally, we assume that the system uncertainties are randomly distributed in the state matrix and the output matrices as well. By decoupling the necessary conditions into a set of matrix inequalities, we show that the search of the observer and the controller gains can be transformed into a convex optimization problem. A set of sufficient linear-matrix-inequality conditions are developed to ensure the existence of a pseudo-state observer-based controller assuring the asymptotic stability of the pseudo-state system under consideration. It is shown that the developed results are less conservative by demonstration of a case study. In the particular case where the system uncertainty are null, sufficient conditions for stability by dynamic output feedback is given when the non-integer differentiation order is between 0 and 2. Detailed proofs are presented and the efficiency of the proposed design is testified by numerical simulations.

2. Preliminaries

Throughout this paper we note by $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{C}$ the set of real number, the set of positive real numbers, and the set of complex numbers, respectively. The notation $A > 0$, with A being a Hermitian matrix (respectively, $A < 0$), means that the matrix A is positive definite (respectively, negative definite). A' is the matrix transpose of A. $X^\star$ stands for the complex conjugate transpose of the matrix X. The notation $\bar{X}$ stands for the matrix conjugate of the complex matrix X. The star element in a given matrix stands for any element that is induced by conjugate transposition. The spec(A) denotes the set of the eigenvalues of the matrix A. We note by I and $\mathbf{0}$ the identity matrix of appropriate dimension and the null matrix of appropriate dimension, respectively. $\Re(Z)$ stands for the real part of the complex matrix/number while $\Im(Z)$ denotes the imaginary part of the complex matrix/number. The notation $[a]_G$ stands for the integer part of the real a. In this paper, Riemann–Liouville fractional differentiation definition is used. Referring to Samko, Kilbas, and Marichev (1987), the fractional integral of a continuously differentiable function $f(t)$ is defined by:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0 \tag{1}$$

where $\alpha \in \mathbb{R}_{>0}$ denotes the fractional-integration order, and

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1}\, dx. \tag{2}$$

The order α fractional derivative of a function $f(t)$, with $\alpha \in \mathbb{R}_{>0}$, is consequently defined by:

$$\begin{aligned} D^\alpha f(t) &= \frac{d^m}{dt^m}\left(I^{m-\alpha} f(t)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau; \quad t > 0 \end{aligned} \tag{3}$$

where m is the smallest integer verifying "$m-1 < \alpha < m$". In Annaby and Mansour (2012), it has been reported that the Riemann–Liouville fractional derivative of order α coincides with the definition of the Grünwald–Letnikov definition, that is

$$D^\alpha_{GL} f(t) = \lim_{T\to 0} \frac{1}{T^\alpha} \sum_{k=0}^{\left[\frac{t}{T}\right]_G} (-1)^k \binom{\alpha}{k} f(t-kT), \quad t > 0, \tag{4}$$

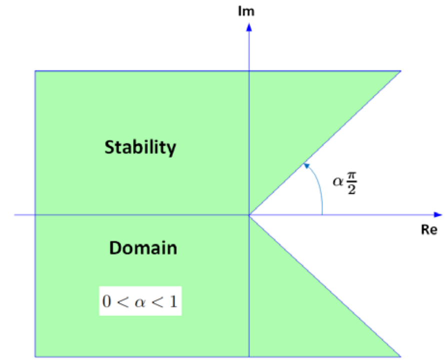

Fig. 1. Stability domain of fractional-order linear systems $0 < \alpha < 1$.

where

$$\binom{\alpha}{k} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, & \text{for; } k \neq 0, \\ 1, & \text{for; } k = 0. \end{cases} \tag{5}$$

The Grünwald–Letnikov definition (4) is a generalization of the ordinary discretization formulas for integer-order derivatives. Depending on the value of the fractional-differentiation order "α", several stability theorems have been stated, see the results in Matignon (1996) for $0 < \alpha < 1$ and Sabatier et al. (2010) for $1 < \alpha < 2$.

Theorem 2.1 (*Moze, Sabatier, & Oustaloup, 2005*)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a real matrix. Then, the fractional-order system:*

$$D^\alpha x(t) = Ax(t), \quad 1 < \alpha < 2, \tag{6}$$

is asymptotically stable, that is, $|\arg(spec(A))| > \alpha\frac{\pi}{2}$ if and only if there exists a symmetric and positive definite matrix P verifying

$$\begin{bmatrix} (AP+PA')\sin(\theta) & (AP-PA')\cos(\theta) \\ \star & (AP+PA')\sin(\theta) \end{bmatrix} < 0 \tag{7}$$

where $\theta = (1-\frac{\alpha}{2})\pi$.

The following result concerns the stabilizability of fractional-order linear systems by means of pseudo-state feedback where its proof is given in Farges et al. (2010). As it has been reported in the literature, the stability of fractional-order linear systems is one particular case of domain stability where the eigenvalues of the system should be located in a specific region of the complex plane as shown in Fig. 1.

Theorem 2.2 (*Farges et al., 2010*)**.** *The fractional-order system: $D^\alpha x(t) = Ax(t) + Bu(t)$, where $0 < \alpha < 1$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$, is stabilizable by pseudo-state feedback $u = Y(rX + \bar{r}\bar{X})^{-1}x$ iff $\exists X = X^\star \in \mathbb{C}^{n\times n} > 0$ and $Y \in \mathbb{R}^{m\times n}$ such that*

$$(rX + \bar{r}\bar{X})'A' + A(rX + \bar{r}\bar{X}) + BY + Y'B' < 0, \tag{8}$$

where $r = e^{i(1-\alpha)\frac{\pi}{2}}$, $i^2 = -1$.

The Schur Complement lemma along with the result of the following lemma are extensively used in the proof of the main statement.

Lemma 2.3 (*Boyd, Ghaoui, Feron, & Balakrishnan, 1994*)**.** *Given real matrices H, E and $Y < 0$ of appropriate dimensions, the inequality: $Y + HF(t)E + E'F'(t)H' < 0$ holds for all $F(t)$ satisfying $F'(t)F(t) \leq I$ if and only if there exists an $\varepsilon > 0$ such that $Y + \varepsilon HH' + \varepsilon^{-1}E'E < 0$.*

Theorem 2.2 will serve as a starting result for further development. The details are given in the following sections.