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Flow functions, control flow functions, and the reach control problem[☆]

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ABSTRACT

The paper studies the reach control problem (RCP) to make trajectories of an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. We introduce the notion of a *flow function*, which provides the analog of a Lyapunov function for the equilibrium stability problem. A flow function comprises a scalar function that decreases along closed-loop trajectories, and its existence is a necessary and sufficient condition for closed-loop trajectories to exit the polytope. It provides an analysis tool for determining if a specific instance of RCP is solved, without the need for calculating the state trajectories of the closed-loop system. Results include a variant of the LaSalle Principle tailored to RCP. An open problem is to identify suitable classes of flow functions. We explore functions of the form $V(x) = \max\{V_i(x)\}$, and we give evidence that these functions arise naturally when RCP is solved using continuous piecewise affine feedbacks. Next we introduce the notion of a *control flow function*. It is shown that the Artstein–Sontag theorem of control Lyapunov functions has direct analogies to RCP via control flow functions.

1. Introduction

We study the reach control problem (RCP) for affine systems on polytopes. The problem is to find a feedback control to make the closed-loop trajectories of an affine system defined on a polytopic state space reach and exit a prespecified facet of the polytope in finite time. The problem has ties to temporal logic specifications (Girard, 2012; Kloetzer & Belta, 2008; Wongpiromsarn, Topcu, & Murray, 2012), and arises in the study of piecewise affine (PWA) hybrid systems consisting of a discrete automaton where each discrete mode is equipped with continuous-time affine dynamics defined on a polytope (Goebel, Sanfelice, & Teel, 2009). When the continuous state crosses a facet of a polytope, the system is transferred to a new discrete mode. The reachability problem for piecewise affine hybrid systems at the continuous level reduces to studying RCP for an affine system on a polytope (Habets, Collins, & van Schuppen, 2006). Interesting applications of RCP can include motion of robots in complex environments (Belta, Isler, & Pappas, 2005), aircraft and underwater vehicles (Belta & Habets, 2006), anesthesia (Ganness, 2010), genetic networks (Belta, Habets, & Kumar, 2002), smart buildings, process control (Haugwitz & Hagander, 2007), among others (Goebel, Sanfelice et al., 2009).

The preponderance of literature on RCP regards simplices because their remarkable structure allows to focus on the essence of the reachability problem (Ashford & Broucke, 2013a,b; Broucke, 2010; Broucke & Ganness, 2014; Habets et al., 2006; Habets & van Schuppen, 2004; Roszak & Broucke, 2006; Semsar Kazerooni & Broucke, 2014). Moreover, the search for feedback classes to solve RCP on simplices is narrowed due to their natural fit with affine feedback (Habets & van Schuppen, 2004). In contrast with simplices, the status for polytopes is more fragmentary. In Habets et al. (2006) a method we call the *simplex-based method* was proposed. In Helwa and Broucke (2011) and Helwa and Broucke (2013) the geometric tools of Broucke (2010) were extended from simplices to polytopes and a variant of RCP called the *monotonic reach control problem* (MRCP) was formulated. The simplex-based method and MRCP are the only known synthesis methods for solving RCP on polytopes (Helwa & Broucke, 2011). It is unlikely that the geometric tools in Broucke and Ganness (2014) and Semsar Kazerooni and Broucke (2014) can be extended to polytopes due to the inherent combinatorial complexity of polytopes. One then turns to numerical approaches. Unfortunately, we encounter examples not solvable by either the simplex-based method or MRCP, yet a PWA

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feedback is numerically obtained and simulations show it solves RCP. This observation sets the stage for this paper.

We require an analysis tool that allows to diagnose rigorously if a candidate PWA (or continuous state) feedback solves RCP, without the need for calculating the state trajectories of the closed-loop system. One immediately recognizes an analogy with Lyapunov analysis for the equilibrium stability problem. But does RCP have an inherent notion of a function that acts like a Lyapunov function? Indeed it does. It was camouflaged as a *flow condition* in Roszak and Broucke (2006). The flow condition is reinterpreted in this paper as a linear scalar function V called a *flow function* that strictly decreases along closed-loop trajectories in the polytope $\mathcal{P}$. Our concept of flow functions appears to be related to so-called density functions used to characterize certain reachability problems (Prajna & Rantzer, 2007) as well as to barrier certificates (Sloth, Wisniewski, & Pappas, 2012).

The contributions of the paper are as follows. In Section 4 we introduce the notion of a flow function. Flow functions provide a necessary and sufficient condition that all trajectories initiated in $\mathcal{P}$ leave it in finite time. In Section 5 we focus on PWA feedback, which is widely used to solve RCP on polytopes (Habets et al., 2006; Helwa & Broucke, 2013). We present results which play the role of converse Lyapunov theorems. The aim is to identify a class of flow functions that naturally emerges when solving RCP by PWA feedback. In Section 6 the analogy with Lyapunov theory is deepened as we explore the Artstein–Sontag theorem for control Lyapunov functions within the context of RCP. Control Lyapunov functions continue to be intensively studied due to important emerging applications in hybrid systems and robotics (Ames, Galloway, & Grizzle, 2012; Goebel, Prieur, & Teel, 2009). We are lead to the notion of *control flow functions*, and we propose a "universal formula" for RCP. These results extend what is a verification tool based on flow functions to a synthesis tool based on control flow functions. A preliminary version of this paper appeared in Helwa and Broucke (2012).

2. Background

We use the following notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, the interior is $\mathcal{K}^\circ$, and the boundary is $\partial\mathcal{K} := \overline{\mathcal{K}} \setminus \mathcal{K}^\circ$, where the notation $\mathcal{K}_1 \setminus \mathcal{K}_2$ denotes elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$. The notation $T_{\mathcal{K}}(x)$ denotes the Bouligand tangent cone to the set $\mathcal{K}$ at a point x (Clarke, Ledyaev, Stern, & Wolenski, 1998). For $x \in \mathbb{R}^n$, $\mathcal{B}_\delta(x)$ denotes the open ball in $\mathbb{R}^n$ centered at x with radius δ. For $x,\ y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation $\mathbf{0}$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\text{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathbb{R}_+$ denotes the set of non-negative real numbers. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be of class $\mathscr{C}^k$ if all its partial derivatives up to order k exist and are continuous. The notation $L_f V(x) = \frac{\partial V}{\partial x} f(x)$ denotes the Lie derivative of $\mathscr{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to function $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions, and let $\phi(t, x_0)$ denote the unique solution of $\dot{x} = f(x)$ starting at x_0. The *upper right Dini derivative* of $V(\phi(t, x_0))$ with respect to t is $D^+V(\phi(t, x_0)) := \limsup_{\tau\to 0^+} \frac{V(\phi(t+\tau,x_0))-V(\phi(t,x_0))}{\tau}$. The *upper Dini derivative of V with respect to f* is given by $D_f^+V(x) := \limsup_{\tau\to 0^+} \frac{V(x+\tau f(x))-V(x)}{\tau}$.

We use some notions from algebraic topology (Munkres, 1996). An n-dimensional simplex $\mathcal{S} := \text{co}\{v_0, \ldots, v_n\}$ is the convex hull of $(n+1)$ affinely independent points $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^n$. A *face* of $\mathcal{S}$ is any simplex spanned by a subset of $\{v_0, \ldots, v_n\}$. A *proper face* of $\mathcal{S}$ is any face of $\mathcal{S}$ different from $\mathcal{S}$ itself. A *facet* of $\mathcal{S}$ is an $(n-1)$-dimensional face. The union of the proper faces of $\mathcal{S}$ is called the boundary of $\mathcal{S}$, denoted $\partial\mathcal{S}$. The interior of $\mathcal{S}$ is $\mathcal{S}^\circ = \mathcal{S} \setminus \partial\mathcal{S}$. An n-dimensional polytope $\mathcal{P} := \text{co}\{v_1, \ldots, v_p\}$ is the convex hull of p points $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ whose affine hull has dimension n. A triangulation $\mathbb{T}$ of an n-dimensional polytope $\mathcal{P}$ is a finite collection of n-dimensional simplices $\mathcal{S}_1, \ldots, \mathcal{S}_L$ such that (i) $\mathcal{P} = \bigcup_{i=1}^{L} \mathcal{S}_i$, (ii) For all $i,\ j \in \{1, \ldots, L\}$ with $i \neq j$, the intersection $\mathcal{S}_i \cap \mathcal{S}_j$ is either empty or a common face of $\mathcal{S}_i$ and $\mathcal{S}_j$. Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$. A point $x \in \mathcal{P}$ lies in the interior of precisely one simplex $\mathcal{S}_x$ in $\mathbb{T}$ whose vertices are, say, $v_1, \ldots, v_k$ (note that $\mathcal{S}_x$ is not necessarily an n-dimensional simplex). Then $x = \sum_{i=1}^{k} \beta_i v_i$, where $\beta_i > 0$ and $\sum_i \beta_i = 1$. Coefficients $\beta_1, \ldots, \beta_k$ are called the *barycentric coordinates* of x. If w is a vertex of $\mathbb{T}$, the *star* of w in $\mathbb{T}$, denoted by $\text{st}(w)$, is the union of the interiors of those simplices in $\mathbb{T}$ that have w as a vertex. It is an open set in $\mathbb{R}^n$. The closure of $\text{st}(w)$, denoted $\overline{\text{st}}(w)$, is called the *closed star* of w in $\mathbb{T}$.

3. Reach control problem

Consider an n-dimensional polytope in $\mathbb{R}^n$, $\mathcal{P} := \text{co}\{v_1, \ldots, v_p\}$ with vertex set $V := \{v_1, \ldots, v_p \mid v_i \in \mathbb{R}^n\}$ and facets $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_r$. The *exit facet* is designated to be the facet $\mathcal{F}_0$ of $\mathcal{P}$. Let h_i be the unit normal to each facet $\mathcal{F}_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J := \{1, \ldots, r\}$, and $J(x) := \{j \in J \mid x \in \mathcal{F}_j\}$. For each $x \in \mathcal{P}$, define the closed, convex cone $\mathcal{C}(x) := \left\{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0,\ j \in J(x)\right\}$. We consider the affine control system defined on $\mathcal{P}$:

$$\dot{x} = Ax + Bu + a, \quad x \in \mathcal{P}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\text{rank}(B) = m$. Let $\mathcal{B} = \text{Im}B$, the image of B. Also, let $\phi_u(t, x_0)$ be the trajectory of (1) under a control law u starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the exit facet $\mathcal{F}_0$ from $\mathcal{P}$ by feedback control.

Problem 3.1 (*Reach Control Problem (RCP)*)**.** Consider system (1) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that: for each $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in \mathcal{F}_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T+\gamma)$.

RCP says that trajectories of (1) starting from initial conditions in $\mathcal{P}$ reach and exit the facet $\mathcal{F}_0$ in finite time, while not first leaving $\mathcal{P}$. Notice that the RCP definition assumes that the dynamics (1) can be extended to a neighborhood of $\mathcal{P}$. A useful shorthand notation is to write $\mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0$ by control $u(x)$ if RCP is solved using $u(x)$.

The class of continuous PWA feedbacks is widely used to solve RCP on polytopes (Habets et al., 2006; Habets & van Schuppen, 2004; Helwa & Broucke, 2013). Let $\mathbb{T}$ be a triangulation of $\mathcal{P}$. Given a state feedback $u(x)$ on $\mathcal{P}$, we say *u is a PWA feedback associated with* $\mathbb{T}$ if for any $x \in \mathcal{P}$, $x = \sum_i \beta_i v_i$ implies $u(x) = \sum_i \beta_i u(v_i)$, where $\{v_i\}$ are the vertices of $\mathcal{S}_x$ and the β_i's are the corresponding barycentric coordinates of x. If $u(x)$ is a PWA feedback associated with $\mathbb{T}$, then for each n-dimensional simplex $\mathcal{S}^k \in \mathbb{T}$, there exist $K_k \in \mathbb{R}^{m\times n}$ and $g_k \in \mathbb{R}^m$ such that u takes the form $u(x) = K_k x + g_k$, $x \in \mathcal{S}^k$. In the literature necessary conditions for a PWA feedback to solve RCP have been identified; they guarantee that closed-loop trajectories only exit $\mathcal{P}$ through $\mathcal{F}_0$ (Habets & van Schuppen, 2004). We say the *invariance conditions are solvable* if for each $x \in \mathcal{P}$, there exists $u \in \mathbb{R}^m$ such that

$$Ax + Bu + a \in \mathcal{C}(x). \tag{2}$$

Solvability of (2) can be checked by solving a linear program at each vertex of $\mathcal{P}$. Once control inputs satisfying (2) are obtained at the vertices, one can apply a straightforward procedure presented in Habets and van Schuppen (2004) to construct a continuous PWA feedback on $\mathcal{P}$ satisfying (2) at all $x \in \mathcal{P}$.