Automatica 55 (2015) 66–72

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On stability analysis methods for large-scale discrete-time systems ☆

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ARTICLE INFO

Article history:
Received 7 February 2013
Received in revised form
19 August 2014
Accepted 14 February 2015

Keywords:
Large-scale systems
Discrete-time systems
Stability analysis
Lyapunov methods

ABSTRACT

This paper proposes a set of Lyapunov-type conditions that are suited for stability analysis of large-scale discrete-time systems. A time-wise relaxation of the Lyapunov function decrease condition is employed to derive a set of global and distributed stability conditions. Essentially, these conditions allow to make a trade-off between complexity and conservatism by extending the time-horizon over which the decrease condition should hold. It is shown that for exponentially stable dynamics and *any* candidate Lyapunov function, there exists a finite time for which the proposed global or distributed stability conditions hold. Hence, it is possible to use functions with a particular structure to make verification of stability scalable for large-scale systems. The developed results are applied to establish stability of a benchmark power systems example.

1. Introduction

Stability analysis of large-scale systems forms an important topic in systems theory and, moreover, it is still an open problem, see, e.g., Michel and Miller (1977); Vidyasagar (1981); Šiljak (1991) and the references therein. Directly analyzing the stability of such systems is generally hampered by the complexity of the overall system. Therefore, to render the stability analysis tractable, the global system is typically decomposed in smaller systems that are interconnected. Analyzing stability of each of the resulting, smaller systems separately, i.e., by neglecting interconnections, is a tractable but highly conservative approach (Šiljak, 1991). Conservativeness can be decreased by employing a set of coupling conditions, which take into account the interconnections. To this end, vector Lyapunov functions (Lakshmikantham, Matrosov, & Sivasundaram, 1991) or the properties of dissipative systems theory can be used.

Dissipative systems theory for interconnected systems, introduced by Willems (1972), establishes that if each system admits a storage function that enjoys a dissipation condition (which takes interconnections into account *via* supply functions) and, moreover, the sum of supply functions is non-positive, then the overall interconnected system is stable. Sufficient conditions for stability of interconnected systems based on this concept were derived by, e.g., Moylan and Hill (1978) and Pota and Moylan (1993). An advantage of this approach is that for linear systems the stability analysis problem can be formulated as a linear matrix inequality (LMI), see, e.g., Langbort, Chandra, and D'Andrea (2004) for a continuous-time framework. The properties of dissipative systems can also be used in small-gain theory where it is shown that the overall interconnected system is stable if the gain functions characterizing the interconnections between systems satisfy a small-gain condition. Early small-gain theorems by Michel and Miller (1977) and Hill and Moylan (1977) made use of linear gain functions while more recently nonlinear gain functions were introduced by Jiang, Teel, and Praly (1994) for two connected systems and generalized to an arbitrary number of systems by Dashkovskiy, Rüffer, and Wirth (2010). Small-gain results are necessary for stability when the interconnection of a system with an arbitrary transfer function that has a norm bound less than or equal to one is considered, see Dahleh and Ohta (1988); Khammash and Pearson (1991). This setting is typically used to analyze robustness of stability. Despite the advances in stability analysis by small-gain and dissipativity theory, both non-conservative and scalable stability analysis tests have not been attained, not even for linear systems. Indeed, small-gain techniques are scalable as they merely require the verification of a, relatively simple, small-gain condition. However, such conditions can be conservative even for very simple linear systems in a state-space representation, as illustrated by an example in Gielen and Lazar (2012). *Positive* linear systems (Rantzer, 2011) are an exception,

☆ The material in this paper was partially presented at the 51st IEEE Conference on Decision and Control, December 10–13, 2012, Maui, Hawaii, USA. This paper was recommended for publication in revised form by Associate Editor Dragan Nesic under the direction of Editor Andrew R. Teel.

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http://dx.doi.org/10.1016/j.automatica.2015.02.034

mainly due to the fact that diagonal Lyapunov functions are non-conservative for this class of linear systems.

This paper reconsiders a time-wise relaxation of the Lyapunov function decrease condition, which was originally proposed by Aeyels and Peuteman (1998) for stability analysis of time-varying systems. Here, the relaxation is used to reduce the conservativeness of a decentralized approach to stability analysis for large-scale systems. Due to this process, the stability criterion is not completely decentralized anymore. As such, the results provide the opportunity to make a trade-off between complexity and conservatism of the stability analysis problem. Moreover, a converse theorem shows that for globally exponentially stable (GES) dynamics and any candidate Lyapunov function, i.e., any function upper and lower bounded by $\mathcal{K}_\infty$ functions, there exists a finite time for which the relaxed stability conditions are satisfied. This opens the door to using candidate functions with particular structure, such as norms, that render verification of stability scalable for large-scale systems. These results are exploited to derive a non-conservative stability analysis theorem for interconnected systems. It is shown that for linear systems, both the global and the decentralized conditions that are derived yield scalable and non-conservative stability tests.

The remainder of the paper is structured as follows. In Section 2, preliminaries are introduced and the conservatism of existing decentralized conditions that make use of storage and supply functions is shown by a simple linear system example. Then, Section 3 presents the main results and their implication for linear dynamics. In Section 4 the developed stability analysis method is applied to a benchmark power systems example. Conclusions are summarized in Section 5.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the real and nonnegative real numbers, respectively. Let $\mathbb{S}^h := \mathbb{S} \times \cdots \times \mathbb{S}$ for any $h \in \mathbb{Z}_{\geq 1}$ denote the h-times Cartesian-product of $\mathbb{S} \subseteq \mathbb{R}^n$. $\|x\|$ is used to denote an arbitrary p-norm of a vector $x \in \mathbb{R}^n$, while $\|x\|_2$ denotes its 2-norm. Let $\{x_l\}_{l\in\{1,\dots,N\}}$, with $x_l \in \mathbb{R}^n$ and $N \in \{1, 2, \dots\}$, denote an arbitrary sequence and define $\text{col}(\{x_l\}_{l\in\{1,\dots,N\}}) := \begin{bmatrix} x_1^\top & \dots & x_N^\top \end{bmatrix}^\top$. For a vector $x \in \mathbb{R}^n$ let $[x]_i$, $i \in \{1, \dots, N\}$ denote the ith entry of x. Let I_n denote the nth dimensional identity matrix and $\mathbf{1}_n$ a column vector with all elements equal to one. Given $A \in \mathbb{R}^{n_1\times n_2}$ and $B \in \mathbb{R}^{m_1\times m_2}$, with $m_1 \geq n_1$ and $m_2 \geq n_2$, let $[B]_{i:i+n_1-1,j:j+n_2-1} = A$ denote that $[B]_{i-1+k,j-1+l} = [A]_{k,l}$ for all $(k, l) \in \{1, \dots, n_1\}\times\{1, \dots, n_2\}$ and for some $(i, j) \in \{1, \dots, m_1 - n_1 + 1\} \times \{1, \dots, m_2 - n_2 + 1\}$, i.e., A is a block in B. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, let $A \succ 0$ and $A \succeq 0$ denote that A is positive definite and positive semi-definite, respectively. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. If $\varphi \in \mathcal{K}$ and additionally $\lim_{s\to\infty} \varphi(s) = \infty$ then φ *belongs to class* $\mathcal{K}_\infty$. A function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \geq 0$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \geq 0$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

2.1. Stability related notions

In this paper, we consider autonomous discrete-time dynamical systems of the form

$$x_{k+1} = G(x_k), \quad k \in \{0, 1, \dots\}, \tag{1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ and n may be a large positive number. Furthermore, we consider that system (1) can be decomposed in a set of $N \geq 2$ interconnected systems of the form

$$x_{(i,k+1)} = g_i(x_{(1,k)}, \dots, x_{(N,k)}), \quad k \in \{0, 1, \dots\}, \tag{2}$$

where $g_i : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N} \to \mathbb{R}^{n_i}$ for all $i \in \{1, \dots, N\}$. The global system dynamics are recovered by letting $n = \sum_{i=1}^N n_i$ and $G(x_k) = \text{col}(\{g_i(x_{(1,k)}, \dots, x_{(N,k)})\}_{i\in\{1,\dots,N\}})$.

Let $\{x_k(\xi)\}_{k\in\{0,1,\dots\}}$ denote the solution of (1) from initial condition $\xi \in \mathbb{R}^n$, i.e., such that $x_0(\xi) := \xi$ and $x_{k+1}(\xi) := G(x_k(\xi))$ for all $k \in \{0, 1, \dots\}$.

Definition 1. System (1) is called *$\mathcal{KL}$-stable* if there exists a function $\beta \in \mathcal{KL}$ such that $\|x_k(\xi)\| \leq \beta(\|\xi\|, k)$ for all $\xi \in \mathbb{R}^n$, $k \in \{0, 1, \dots\}$. If in addition $\beta(s, k) := c\mu^k s$ for some $c \in \mathbb{R}$, $c \geq 1$ and $\mu \in [0, 1)$, then system (1) is called *globally exponentially stable* (GES). □

Definition 2. A map $G : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathcal{K}$-bounded if there exists $\sigma_G \in \mathcal{K}$ such that $\|G(x)\| \leq \sigma_G(\|x\|)$ for all $x \in \mathbb{R}^n$. If $\sigma_G(s) := L_G s$ for some $L_G \in \mathbb{R}$, $L_G > 0$ then G is called globally Lipschitz-bounded. □

Note that $\mathcal{KL}$-stable as defined here is a global property. Observe that the $\mathcal{K}$-bounded property does not imply continuity of the map G, except at $x = 0$, which is a necessary condition for Lyapunov stability.

Fact 3. *Suppose that system (1) is $\mathcal{KL}$-stable. Then the map G that describes the system dynamics is $\mathcal{K}$-bounded. Moreover, if system (1) is GES, then G is globally Lipschitz-bounded.*

Proof. By the definition of $\mathcal{KL}$-stability it holds that $\|x_1(\xi)\| = \|G(\xi)\| \leq \beta(\|\xi\|, 1)$. Hence, G is $\mathcal{K}$-bounded with $\sigma_G(s) := \beta(s, 1) \in \mathcal{K}$. Furthermore, by the definition of GES, G is globally Lipschitz-bounded with $L_G := c\mu > 0$. ■

Fact 3 demonstrates that the $\mathcal{K}$-bounded (globally Lipschitz-bounded) property is not restrictive for analysis of $\mathcal{KL}$-stability (GES). Note also that if the $\mathcal{K}$-bounded property in Definition 2 holds with a $\sigma_G \in \mathcal{K}$, then it also holds with some $\bar{\sigma}_G \in \mathcal{K}_\infty$.

2.2. A motivating example

If the state–space dimension is large, construction of a Lyapunov function for the global system (1) is not practicable. That is why distributed stability tests are pursued instead. In this section we demonstrate that, however, existing results that are based on the assumption that interconnections between subsystems are dissipative come with conservatism, even for very simple linear systems. To this end, a result for stability analysis of discrete-time interconnected dissipative systems of the form (2), which stems from the pioneering article by Willems (1972, see page 15), is presented next. This result is an analogy of continuous-time results, see, e.g., Theorem 1 in Langbort et al. (2004).

Theorem 4. *Suppose that there exist storage and supply functions $\{W_i, S_{i,j}\}_{(i,j)\in\{1,\dots,N\}^2}$, with $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $S_{i,j} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, a constant $\rho \in [0, 1)$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $i \in \{1, \dots, N\}$*

$$\alpha_1(\|\xi_i\|) \leq W_i(\xi_i) \leq \alpha_2(\|\xi_i\|), \tag{3a}$$

$$W_i(g_i(\xi_1, \dots, \xi_N)) \leq \rho W_i(\xi_i) + \sum_{j=1}^N S_{i,j}(\xi_i, \xi_j), \tag{3b}$$

$$\sum_{j=1}^N \sum_{l=1}^N S_{j,l}(\xi_j, \xi_l) \leq 0, \tag{3c}$$

for all $\xi := \text{col}(\{\xi_i\}_{i\in\{1,\dots,N\}}) \in \mathbb{R}^n$. Then, system (1) is $\mathcal{KL}$-stable. Moreover, if (3a) holds with $\alpha_1(s) := c_1 s^\lambda$ and $\alpha_2(s) := c_2 s^\lambda$ for some $(c_1, c_2) \in \mathbb{R}^2$, $0 < c_1 \leq c_2$ and $\lambda \in \{1, 2, \dots\}$, then system (1) is GES.