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Brief paper

A fundamental control limitation for linear positive systems with application to Type 1 diabetes treatment[☆]

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ABSTRACT

This paper presents a fundamental design trade-off applicable to a class of linear positive systems. The result connects the maximum and minimum output response peaks due to a disturbance for all feasible inputs. A core consequence of the result is that, when the disturbance pulse response peaks faster than the input pulse response, then attempts to minimise the maximum peak response to the disturbance are necessarily accompanied by unavoidable undershoot at a later time. The result has potential application in many areas. For example, it provides a defensible benchmark for comparison of all possible insulin treatment strategies for Type 1 diabetes patients. This includes any control strategy implemented in the context of an Artificial Pancreas (AP). The lack of such a benchmark has been a deficiency in the existing AP literature.

1. Introduction

Positive systems are systems where all state variables remain non-negative at all times (Berman, Neumann, & Stern, 1989; Krasnosel'skij, Lifshits, & Sobolev, 1989). Many real world problems in engineering, economics and biology can be modelled as positive systems (Farina & Rinaldi, 2011; Kaczorek, 2002).

In the current paper we present an unavoidable trade-off in the control system design for a class of positive systems. The result presented here falls under the broad classification of fundamental control limitations (Doyle & Stein, 1981; Goodwin, Graebe, & Salgado, 2001; Horowitz, 1963; Seron, Braslavsky, & Goodwin, 1997). This topic has been a cornerstone problem in modern control theory. Known results include the much celebrated Bode Sensitivity integral trade-off (Bode, 1945). Results are available for both linear and nonlinear systems and include both time and frequency domain behaviour (Chen, 1995; Freudenberg, Hollot, Middleton, & Toochinda, 2003; Freudenberg & Looze, 1985; Middleton, 1991). However, to the best of the authors' knowledge, there has been no previous trade-off enunciated for positive systems.

An illustration of the implications and usefulness of the result developed here is given in the context of blood glucose regulation for Type 1 diabetes patients. Type 1 diabetes is a major health issue. Almost 1% of people in the western world suffer from this disease. Treatment is invasive and disruptive to the patient. It is a quintessential example of positive systems since all variables, e.g. blood glucose and insulin concentration, are restricted to take positive values.

Because of its impact, much has been written on the topic of blood glucose regulation. Indeed, there has been a major effort directed at the development of an artificial pancreas aimed at automating the control of blood glucose level (Bequette, 2005; Harvey, Wang, & Grosman, 2010; Klonoff, Cobelli, Kovatchev, & Zisser, 2009; Lee, Buckingham, Wilson, & Bequette, 2009). Although huge advances have been made, it is generally agreed that a fully automated system remains some distance away. The result presented in the current paper is believed to provide important insights into the development of an Artificial Pancreas and, more broadly, for the treatment of Type 1 diabetes patients.

The layout of the remaining part of the paper is as follows: in Section 2 we summarise relevant system theory and describe the model. Section 3 develops the main result of the paper. Section 4

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briefly discusses the implications of the result in the treatment of Type 1 diabetes. Conclusions are drawn in Section 5.

2. System theory background

We consider a linear dynamic single-input single-output time-invariant system acted upon by an external disturbance. The system is assumed to belong to the class of positive systems in which all variables are positive for all time.

We assume that the system operates around some steady state value $(\bar{y}_s, \bar{u}_s)$ of the output and input, respectively. We also assume the existence of a positive endogenous input $\bar{E}_s$.

The incremental output response to a unit disturbance input is modelled by the pulse response h_t^d. Similarly, the incremental output response to a unit input pulse is modelled by h_t^u. For simplicity, we assume that h_t^d and h_t^u are both zero for all $t \geq N$.

Then, via the principle of superposition, the response at time t due to a disturbance sequence $\{d_j;\ j = 0, 1, \ldots\}$ and to an input sequence $\{u_j;\ j = 0, 1, \ldots\}$ is

$$y_t = \bar{E}_s - \sum_{i=1}^{N} h_i^u u_{t-i} + \sum_{i=1}^{N} h_i^d d_{t-i}. \tag{1}$$

In circumstances where no disturbances occur and the input is set to $\bar{u}_s$, then

$$\bar{y}_s = \bar{E}_s - \bar{u}_s \sum_{i=1}^{N} h_i^u. \tag{2}$$

Remark 1. In this paper we will not focus on how a model of the form of (1) can be obtained experimentally. Indeed, the result presented here depends only upon the existence of the model and not on knowledge of the model. Of course, a practical implementation of the gold standard presented in the sequel would require an estimate of the model. In this context, a sensitivity analysis to model estimation errors lies outside the scope of the paper.

3. A fundamental trade-off

To create a fundamental trade-off, we consider a disturbance pulse of intensity $\bar{D}$ applied at $t = 0$ and determine bounds on the subsequent overshoot, i.e. a transition over the setpoint, and undershoot, i.e. a transition below the setpoint, in the response for *all possible* causal control strategies.

Let us assume that some control scenario leads to the input sequence $u_{t-N}, \ldots, u_0, u_1, \ldots, u_{t-1}$. Then, from (1), the corresponding output response at time t is given by

$$y_t = \bar{E}_s + \bar{D}h_t^d - \sum_{j=1}^{N} h_j^u u_{t-j}, \quad t \geq 1. \tag{3}$$

We focus our attention on two specific time samples, namely T_1 and $T_2 > T_1$, which are arbitrarily chosen and are illustrated schematically in Fig. 1. Note that $M_1,\ M_2$ are the values of the disturbance response, h_t^d, at times $t = T_1$ and $t = T_2$.

In the sequel, it will be helpful to work with input increments around the steady input $\bar{u}_s$. Thus, we introduce

$$\tilde{u}_j = u_j - \bar{u}_s \tag{4}$$

where, due to causality of the strategy, $\tilde{u}_j = 0$ for $j < 0$. We can then rewrite (3) as

$$\begin{aligned} y_t &= \bar{E}_s + \bar{D}h_t^d - \sum_{j=1}^{N} h_j^u [u_{t-j} - \bar{u}_s + \bar{u}_s] \\ &= \left[\bar{E}_s - \sum_{j=1}^{N} h_j^u \bar{u}_s\right] + \bar{D}h_t^d - \sum_{j=1}^{N} h_j^u \tilde{u}_{t-j} \\ &= \bar{y}_s + \bar{D}h_t^d - \sum_{j=1}^{N} h_j^u \tilde{u}_{t-j} \end{aligned} \tag{5}$$

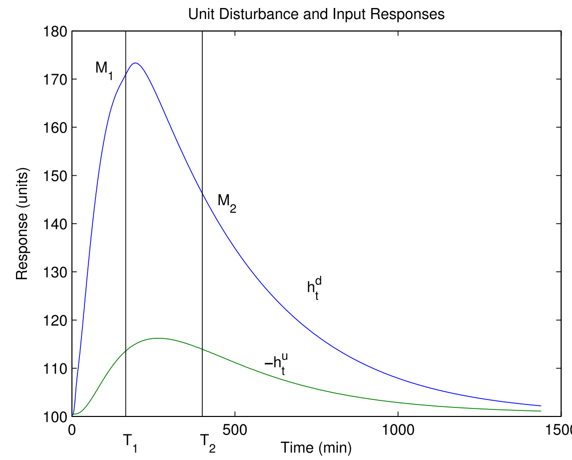

Fig. 1. Input and disturbance pulse responses.

where we have used (2). We also constrain the input to be positive and hence

$$\tilde{u}_j \geq -\bar{u}_s. \tag{6}$$

With the above as background, we have the fundamental law presented in Theorem 2. The result holds under general conditions, but has particular significance when the disturbance pulse response peaks faster than the input pulse response. In the latter case, the result shows that all attempts to diminish the effect of a disturbance in the output at an early time, necessarily lead to undershoot at a later time. In addition, we derive an upper bound on the lowest excursion and show that the bound is achievable by a particular control strategy involving a single input pulse.

Theorem 2. *Let a disturbance pulse of intensity $\bar{D}$ be applied at $t = 0$. Let $y_{T_1},\ y_{T_2}$ be the output response at $t = T_1$ and at $t = T_2$, respectively, then the following statements hold:*

(a) *Consider a fixed value of y_{T_1}, then $\bar{y}_{T_2}$ is an upper bound on y_{T_2}, i.e. $y_{T_2} \leq \bar{y}_{T_2}$ and is given by*

$$\bar{y}_{T_2} = C_1 + C_2\bar{D} + r^* y_{T_1} \tag{7}$$

where

$$C_1 = \bar{y}_s + D_3 - r^*\bar{y}_s + D_2 - r^* D_1 \tag{8}$$

$$C_2 = M_2 - r^* M_1 \tag{9}$$

and where $\bar{y}_s$ is the steady state value of y_t when $\bar{u}_s$ is applied (see (2)). Additionally,

$$r^* = \min_{k \in [0, T_1]} \left[\frac{h_{T_2-k}^u}{h_{T_1-k}^u}\right] \tag{10}$$

$$D_1 = \sum_{j=1}^{T_1-1} h_{T_1-j}^u \bar{u}_s \tag{11}$$

$$D_2 = \sum_{j=1}^{T_1-1} h_{T_2-j}^u \bar{u}_s \tag{12}$$

$$D_3 = \sum_{j=T_1}^{T_2} h_{T_2-j}^u \bar{u}_s \tag{13}$$

and $M_1,\ M_2$ are the values of the disturbance pulse response, h_t^d, at times $t = T_1$ and $t = T_2$, respectively, in the absence of additional input signals (see Fig. 1) and where $1 < T_1 < T_2$.