Automatica 55 (2015) 132–139

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Model reduction for delay differential equations with guaranteed stability and error bound[☆]

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ARTICLE INFO

Article history:
Received 5 May 2014
Received in revised form
6 January 2015
Accepted 9 February 2015
Available online 24 March 2015

Keywords:
Model reduction
Delay differential equations
Stability
Error bound
Nonlinearity
Time-varying delays

ABSTRACT

In this paper, a structure-preserving model reduction approach for a class of delay differential equations is proposed. Benefits of this approach are, firstly, the fact that the delay nature of the system is preserved after reduction, secondly, that input–output stability properties are preserved and, thirdly, that a computable error bound reflecting the accuracy of the reduction is provided. These results are applicable to large-scale linear delay differential equations with constant delays, but also extensions to a class of nonlinear delay differential equations with time-varying delays are presented. The effectiveness of the results is evidenced by means of an illustrative example.

1. Introduction

Complex dynamical system models in terms of delay differential equations appear naturally in a wide variety of problems in for example engineering, biology and control theory (Altintas, 2000; Erneux, 2009; Gu, Kharitonov, & Chen, 2003; Michiels & Niculescu, 2007; Stepan, 1989). In support of the dynamic analysis, optimization or controller design for such systems, we often desire to employ methods for model complexity reduction. Model order reduction is a tool for the order reduction of high-order dynamical systems in pursuit of complexity reduction. A wide range of results are available for the model order reduction of models in terms of ordinary differential equations, see e.g. Antoulas (2005), Bai (2002), Craig (2000), deKlerk, Rixen, and Voormeeren (2008), Freund (2003), Gallivan, Grimme, and Van Dooren (1999) and Gugercin and Antoulas (2004).

Also for delay differential equations (DDEs) different approaches for model reduction are available, albeit to a more limited extent. Methods for the finite-dimensional approximation of delay systems through rational approximations have been proposed in Mäkilä and Partington (1999a,b), see also Glover, Curtain, and Partington (1988). Recently, a technique based on the dominant pole algorithm has been proposed to obtain a rational approximation of an input–output transfer function representing second-order delay differential equations (Saadvandi, Meerbergen, & Jarlebring, 2012). A Krylov-based model reduction approach leading to finite-dimensional (delay-free) model approximations has been proposed in Michiels, Jarlebring, and Meerbergen (2011). In Harkort and Deutscher (2011), Krylov methods for infinite-dimensional systems, applicable to delay systems, have been proposed also leading to finite-dimensional approximations. The above methods have the common property that the resulting models are of a finite-dimensional nature; hence the inherent delay nature of the original system is lost.

In this paper, we aim at constructing reduced-order models which preserve the delay nature of the system dynamics (i.e. the reduced-order model is also a delay differential equation, though of a reduced order). The desire to preserve the delay nature in the reduced-order model is motivated by, firstly, the fact that, for a given order of the reduced model, a reduced model in the form of

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Thomas Meurer under the direction of Editor Miroslav Krstic.

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http://dx.doi.org/10.1016/j.automatica.2015.02.031
0005-1098/

a delay differential equation is in general more accurate than a reduced model in the form of a delay free system, see e.g. Saadvandi et al. (2012), and, secondly, the fact that by preserving the delay nature also related system properties (such as e.g. the infinite-dimensional system character and the infinite number of eigenvalues) are preserved. Such structure-preserving model reduction techniques for delay differential equations, yielding reduced-order delay models, are needed as, on the one hand, powerful simulation and controller synthesis techniques for such systems have become available in the recent past (Bellen, Maset, Zennaro, & Guglielmi, 2009; Gu et al., 2003; Michiels & Niculescu, 2007; Shampine & Thompson, 2001), while, on the other hand, the main bottleneck of these methods is that in most cases they require the order of the delay differential equation to be moderate. In Beattie and Gugercin (2009), interpolatory projection methods based have been proposed, which are also applicable to delay systems and preserve the delay nature in the reduced-order model. In Jarlebring, Damm, and Michiels (2013), a structure preserving model reduction technique for delay differential equations has been proposed, which extends the notion of position balancing from second-order systems to time-delay systems and relies on solving delay Lyapunov equations (Kharitonov, 2013).

In this paper, we propose a structure-preserving model order reduction strategy for a class of delay differential equations, based on balancing techniques, which, firstly, preserves the delay nature of the model, secondly, guarantees the preservation of both internal and input–output stability properties and, thirdly, comes with a computable error bound on the reduced-order model. We note that the latter two aspects (stability preservation and an error bound) are lacking in the existing results in the literature mentioned above. Error bounds have been proposed for finite-dimensional rational approximations, see Glover et al. (1988). Moreover, error bounds and the preservation of stability are also guaranteed in the works (Lam, Gao, & Wang, 2005; Xu, Lam, Huang, & Yang, 2001), in which an H_∞ model reduction approach for linear time-delay systems has been proposed.

The benefits of the approach proposed in the current paper in comparison with the approach in Lam et al. (2005) and Xu et al. (2001) are twofold. Firstly, by the grace of the fact that we employ balancing-type techniques as a basis, which use the solution to two algebraic Lyapunov equations, the approach proposed here is applicable to systems up to order $O(10^3)$ using standard (Bartels–Stewart) algorithms and to systems up to order $O(10^6)$ using tailored algorithms, see e.g. Benner and Saak (2013). On the other hand, the approach in Lam et al. (2005) and Xu et al. (2001) of reformulating the model reduction problem as a H_∞-norm minimization problem of the 'error system', induced by the reduction, leads to an (non-convex) optimization problem constrained by a set of matrix inequalities. The latter fact makes such an approach more computationally complex and hence obstructs applicability to systems of high order. Secondly, we propose a natural approach of decomposing the delay system dynamics in terms a feedback interconnection between a finite-dimensional linear part and a delay-operator part. This approach is natural in many applications, in which the delay only affects certain outputs, see e.g. models for high-speed milling processes (Altintas, 2000; Faassen, van de Wouw, Oosterling, & Nijmeijer, 2003; Insperger & Stepan, 2000) and drilling processes (Germay, Denoel, & Detournay, 2009; Germay, van de Wouw, Sepulchre, & Nijmeijer, 2009). Moreover, such a decomposition allows to employ incremental $\mathcal{L}_2$-gain properties of the systems in the feedback interconnection to guarantee the preservation of stability and to provide an error bound. The latter analysis strategy is also instrumental in supporting the extension of the model reduction approach to systems with nonlinearities and (uncertain) time-varying delays. Finally, we provide an expression for an a priori error bound depending on (1) the properties of the high-order system, (2) the delay and (3) the order of the reduced-order system.

The structure of the paper is as follows. Section 2 specifies in detail the problem formulation and the class of delay systems considered. Next, in Section 3 the model reduction approach is introduced as applicable to a class of linear delay differential equations with constant delays. Section 4 presents the results on the preservation of stability properties and a bound on the reduction error. Moreover, in this section also the extension to nonlinear systems with time-varying delays is highlighted. Finally, Section 5 presents an illustrative example and Section 6 presents concluding remarks.

Notation. The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, $|x|^2 = x^T x$. The space $\mathcal{L}_2^n$ consists of all functions $x : [0, \infty) \to \mathbb{R}^n$ which are bounded using the norm $\|x\|_2^2 := \int_0^\infty |x(t)|^2 dt$.

2. Problem formulation

Consider a generic class of linear delay differential equations (with point-wise delay) that can be formulated in the following form:

$$\Sigma : \begin{cases} \dot{x}(t) = \bar{A}_0 x(t) + \bar{A}_1 x(t-\tau) + Bu(t), \\ y(t) = C_y x(t) + D_{yu} u(t) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^p$. Alternatively, the dynamics in (1) can be written in the following form, to be used in the remainder of this paper:

$$\Sigma : \begin{cases} \dot{x}(t) = A_0 x(t) + A_1 (x(t) - x(t-\tau)) + Bu(t), \\ y(t) = C_y x(t) + D_{yu} u(t) \end{cases} \tag{2}$$

with $A_0 = \bar{A}_0 + \bar{A}_1$ and $A_1 = -\bar{A}_1$.

We study the problem of model reduction for delay differential equations of the form (2) and later comment on extensions to certain classes of nonlinear systems and the case of (uncertain) time-varying delays. Let us explicate what we mean by model reduction for a delay differential equation as in (2). Hereto, we recall the fact that the model in (2) is infinite-dimensional, i.e. the initial condition for system (2) is the function segment $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ to $\mathbb{R}^n$. In fact, we aim to preserve the infinite-dimensional nature of the system in the model reduction approach to be proposed. Still, we can speak of the order of the delay differential equation (2) in terms of the number of equations in the first equality in (2), which in this case is n. Now, we aim at constructing a reduced-order model in terms of a linear delay differential equation of order $\hat{n}$ (i.e. with 'state' $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$) such that,

- the reduced-order model is also a delay differential equation similar in form to (2), i.e. the delay-nature of the system is preserved;
- $\hat{n} < n$, i.e. model (order) reduction is achieved;
- if (2) is asymptotically stable (for $u = 0$) and hence finite $\mathcal{L}_2$-gain stable with respect to the input/output pair (u, y), then the reduced-order model is also asymptotically stable (for $u = 0$) and $\mathcal{L}_2$-gain stable with respect to the same input/output pair $(u, \hat{y})$, where $\hat{y}$ is the output of the reduced-order system;
- there exists a computable error bound reflecting the accuracy of the reduction.

Clearly, in the above problem statement we aim at the preservation of asymptotic stability for zero inputs[2] and $\mathcal{L}_2$-gain stability with respect to the input/output pair (u, y), the latter of which is defined below (see also Fridman & Shaked, 2006).

[2] For a definition of asymptotic stability for functional differential equations, we refer to Gu et al. (2003) and Hale and Verduyn Lunel (1993).