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Brief paper

Mean–variance portfolio selection in a complete market with unbounded random coefficients☆

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ABSTRACT

This paper concerns a mean–variance portfolio selection problem in a complete market with unbounded random coefficients. In particular, we use adapted processes to model market coefficients, and assume that only the interest rate is bounded, while the appreciation rate, volatility and market price of risk are unbounded. Under an exponential integrability assumption of the market price of risk process, we first prove the uniqueness and existence of solutions to two backward stochastic differential equations with unbounded coefficients. Then we apply the stochastic linear–quadratic control theory and the Lagrangian method to solve the problem. We represent the efficient portfolio and efficient frontier in terms of the unique solutions to the two backward stochastic differential equations. To illustrate our results, we derive explicit expressions for the efficient portfolio and efficient frontier in one example with Markovian models of a bounded interest rate and an unbounded market price of risk.

1. Introduction

In Markowitz (1952), the mean–variance portfolio selection problem was initially formulated and discussed in a discrete-time single-period setting. Markowitz's mean–variance model provided a simple solution to the trade-off between the two most important aspects of investment, namely, risk and return. In the last several decades, mean–variance portfolio selection has become an important research topic and many efforts have been made to extend Markowitz's mean–variance model from the static setting to the dynamic one. Li and Ng (2000) and Zhou and Li (2000) were the first to obtain analytical solutions to the multi-period and the continuous-time mean–variance portfolio selection problems, respectively, where an embedding technique was adopted in both papers to deal with the time-inconsistent term in the cost functional and to locate the solution of the original problem via that of an auxiliary problem. A key feature of Li and Ng (2000) and Zhou and Li (2000) is that model coefficients are assumed to be deterministic or constant. However, many empirical studies have revealed that interest rate, appreciation rate, and volatility are random rather than deterministic. For this reason, it is not unreasonable to relax the assumption of deterministic/constant coefficients in mean–variance problems.

To investigate mean–variance portfolio selection problems with random coefficients, the backward stochastic differential equation (BSDE) approach has been found to be particularly useful. See, for example, Hu, Jin, and Zhou (2012), Hu and Zhou (2005), Lim (2004, 2005), and Lim and Zhou (2002). In this series of papers, the continuous-time mean–variance problems were formulated in a random environment with model coefficients given by bounded stochastic processes. On the one hand, such a random environment is general, in which model coefficients are not necessarily Markovian with respect to the underlying filtration. But on the other hand, the boundedness assumption of coefficients is restrictive since most well-known and tractable models for stochastic interest rate, appreciation rate and volatility, if not all, induce unbounded coefficients. So the boundedness assumption may limit applications of aforementioned papers in concrete examples, such as the CEV model (Cox, 1975), the Hull–White model (Hull & White, 1987), the Heston model (Heston, 1993), where stochastic volatility/variance processes are obviously unbounded. Although recently there has been some interest in solving mean–variance portfolio selection problems with unbounded coefficients driven by specific Markovian models (see Chiu & Wong, 2011; Shen & Zeng, 2015; Shen, Zhang, & Siu, 2014, and among others), a unified framework to include the general non-Markovian models is lacking. This paper aims at providing a solution to this problem.

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In this paper, we consider a continuous-time mean–variance portfolio selection problem in a complete market consisting of a risk-free asset and multiple risky assets. We assume that the interest rate, appreciation rate, volatility and market price of risk are given by adapted processes. In contrast to the previous literature, we assume that only the interest rate process is bounded and allow others to be unbounded. This adds technical difficulties in the uniqueness and existence of solutions to relevant BSDEs. We overcome the technical difficulties by imposing an exponential integrability assumption on the market price of risk process. Using measure change techniques, we prove that a linear BSDE and a backward stochastic Riccati equation (BSRE) have unique solutions in proper spaces. To solve the mean–variance problem, we first apply the stochastic linear–quadratic control theory to solve a quadratic-loss minimization problem, whose solution is given by the unique solutions to the linear BSDE and BSRE. By the Lagrangian method, we derive the efficient portfolio and efficient frontier of the mean–variance problem. To illustrate our results, we discuss one example with a bounded random interest rate and an unbounded random market price of risk, both of which are given by Markovian models. Using the theory of matrix-valued Riccati equations and Sturm–Liouville equations, we get explicit expressions for the efficient portfolio and efficient frontier for the example in terms of matrix exponentials and spectral expansions.

Our paper differentiates with the existing literature (Lim, 2004; Lim & Zhou, 2002) mainly in four aspects. First of all, market coefficients in our paper are unbounded, while those in Lim (2004) and Lim and Zhou (2002) are bounded. This is the main difference between our paper and the existing literature and causes the subsequent differences. Due to the unboundedness of random coefficients, we have to deal with BSDEs with unbounded coefficients, which are less well-studied in the literature. Therefore, the generalization to unbounded random coefficients is by no means trivial. Second, the solvability of relevant BSDEs and the method we adopt to prove the solvability are different from those in Lim (2004) and Lim and Zhou (2002). Our paper proves that relevant BSDEs have unique solutions in $\mathcal{L}^p$, while the solution spaces of BSDEs obtained in Lim (2004) and Lim and Zhou (2002) are $\mathcal{L}^2$. Moreover, the proof in our paper is accomplished in two steps. We first construct some new probability measures equivalent to the real-world measure, and prove that some auxiliary BSDEs under these equivalent measures admit unique solutions in $\mathcal{L}^{2p}$. Then we identify relations between the unique solutions to auxiliary BSDEs and original BSDEs, and rely on the exponential integrability assumption and some fundamental inequalities to confirm that the unique solutions to original BSDEs indeed live in $\mathcal{L}^p$ under the real-world measure. This is different from Lim and Zhou (2002), where the uniqueness and existence of solutions to BSDEs are obtained under the real-world measure directly. Although measure change techniques were also used to prove the solvability of a linear BSDE in Lim (2004), its solution space under the real-world measure was not discussed explicitly (see Theorem 4.2 in Lim, 2004). Our paper may shed light on this gap. Third, different from Lim (2004) and Lim and Zhou (2002), we define the admissible set as a space in which the products of portfolio strategies and some market coefficients are (square) integrable. If coefficients were assumed to be bounded as in Lim (2004) and Lim and Zhou (2002), then they might have no impact on the admissibility of portfolio strategies. However, the unbounded coefficients do have impacts on the space of portfolio strategies in our paper. This leads us to modify the definition of admissible strategies and seek a different approach to verifying the admissibility of the obtained efficient portfolio. The fourth difference lies in the condition, under which the mean–variance problem is feasible. The portfolios constructed in Lim (2004) and Lim and Zhou (2002) for proving the feasibility cannot be used in our paper, since they are not admissible in our unboundedness framework. We construct a different portfolio and obtain a different condition for feasibility (refer to Lemma 4.1). The main contribution of this paper is the introduction of the exponential integrability assumption for the market price of risk, which makes the mean–variance problem with unbounded random coefficients well-defined and rigorously solvable. This assumption allows us to investigate BSDEs with unbounded random coefficients, and use the well-posed solutions for relevant BSDEs to tackle the mean–variance problem with unbounded random appreciation rate, volatility and market price of risk. See also Shen and Wei (2014) for the investment, consumption and insurance problem with unbounded random parameters, where a stronger exponential integrability condition was imposed.

The rest of this paper is organized as follows. Section 2 introduces the basic model dynamics and assumptions, and states the mean–variance portfolio selection problem. In Section 3, the uniqueness and existence of solutions to two BSDEs with unbounded coefficients are discussed. In Section 4, the efficient portfolio and efficient frontier of the mean–variance problem are derived. Section 5 is devoted to solving one example. Section 6 concludes the paper.

2. Problem formulation

In this section, we first introduce notation, model dynamics and assumptions to be used throughout the paper. Then we formulate the mean–variance portfolio selection problem in a random environment with unbounded coefficients.

Let $[0, T]$ be a finite time horizon, where $T < \infty$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which is defined an n-dimensional, standard Brownian motion $\{W(t)\}_{t\in[0,T]} := \{(W_1(t), W_2(t), \ldots, W_n(t))^\top\}_{t\in[0,T]}$. We further equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a right-continuous, $\mathbb{P}$-complete filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the Brownian motion $W(\cdot)$. Here $\mathbb{P}$ is a real-world probability measure from which a family of equivalent probability measures can be generated. Denote by $\mathbb{E}[\cdot]$ the expectation under $\mathbb{P}$ and by $\mathbb{E}_t[\cdot]$ the conditional expectation under $\mathbb{P}$ given $\mathcal{F}_t$. For any $k, l \geq 1$ and any integer $m \geq 1$ and under $\mathbb{P}$, we denote the space of all $\mathcal{F}_T$-measurable random variables $\xi : \Omega \mapsto \mathbb{R}^m$ such that $\mathbb{E}[|\xi|^k] < \infty$ by $\mathcal{L}^k_{\mathcal{F}_T,\mathbb{P}}(\Omega; \mathbb{R}^m)$, the space of all $\mathbb{R}^m$-valued $\mathbb{F}$-adapted processes $\{f(t)\}_{t\in[0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}[\sup_{0\le t\le T} |f(t)|^k] < \infty$ by $\mathcal{L}^k_{\mathbb{F},\mathbb{P}}(0, T; \mathcal{C}(0, T; \mathbb{R}^m))$, the space of all $\mathbb{R}^m$-valued $\mathbb{F}$-adapted essentially bounded processes with $\mathbb{P}$-a.s. continuous sample paths by $\mathcal{L}^\infty_{\mathbb{F},\mathbb{P}}(0, T; \mathcal{C}(0, T; \mathbb{R}^m))$, the space of all $\mathbb{R}^m$-valued $\mathbb{F}$-adapted processes $\{f(t)\}_{t\in[0,T]}$ such that $\mathbb{E}[(\int_0^T |f(t)|^2 dt)^{\frac{k}{2}}] < \infty$ by $\mathcal{L}^k_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}^m)$, and the space of all $\mathbb{R}^m$-valued $\mathbb{F}$-adapted processes $\{f(t)\}_{t\in[0,T]}$ such that $\mathbb{E}[(\int_0^T |f(t)|^l dt)^{\frac{k}{l}}] < \infty$ by $\mathcal{L}^{l,k}_{\mathbb{F},\mathbb{P}}(0, T; \mathbb{R}^m)$. With $\mathbb{P}$ replaced by other probability measures, we can define similar spaces under other probability measures. Throughout this paper, we let $K, \delta > 0$ and $p, q > 1$ be four generic constants, which may be different from line to line.

Consider a complete market consisting of $n+1$ primitive assets, including one risk-free asset and n risky assets. The risk-free asset is a bank account and its price process $\{S_0(t)\}_{t\in[0,T]}$ evolves as

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1, \tag{1}$$

where $r(t)$ is the risk-free interest rate for instantaneous borrowing and lending at time t. The other n risky assets are stocks and their dynamics $\{S_i(t)\}_{t\in[0,T]}$, for $i = 1, 2, \ldots, n$, are governed by the following SDEs:

$$\begin{aligned} dS_i(t) &= S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\right], \\ S_i(0) &= s_i > 0, \end{aligned} \tag{2}$$