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Brief paper

Zonotopes and Kalman observers: Gain optimality under distinct uncertainty paradigms and robust convergence ☆

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ABSTRACT

State bounding observation based on zonotopes is the subject of this paper. Dealing with zonotopes is motivated by set operations resulting in simple matrix calculations with regard to the often huge number of facets and vertices of the equivalent polytopes. Discrete-time LTV/LPV systems with state and measurement uncertainties are considered. Based on a new zonotope size criterion called F_W-radius, and by merging optimal and robust observer gain designs, a Zonotopic Kalman Filter (ZKF) is proposed with a proof of robust convergence. The notion of covariation is introduced and results in an explicit bridge between the zonotopic set-membership and the stochastic paradigms for Kalman Filtering. No intersection is used and the influence of the reduction operator limiting to a tunable maximum the size of the matrices involved in the zonotopic set computations is fully taken into account in the LMI-based robust stability analysis. A numerical example illustrates the effectiveness of the proposed ZKF.

1. Introduction

State observation is often a key step for the satisfaction of advanced monitoring and/or control requirements in many engineering applications. Explicitly characterizing how the model and measurement uncertainties can influence the possible state values is also useful, not only when an automated decision-making is further required (e.g. fault diagnosis Ding, 2008), but also in some control frameworks (Efimov, Raïssi, & Zolghadri, 2013). Two paradigms can be used to model uncertainties: The *stochastic* one relies on probability theory and mainly deals with random variables. Usually, assumptions about their probability distribution are required. For instance, state estimators based on Kalman filtering (Kalman, 1960; Lewis, 1986; Maybeck, 1979; Sorenson, 1983) rely on covariance matrices to model usually Gaussian state and measurement random perturbations. Though well suited to take the distribution of random noises into account, this modeling of uncertainty may be less representative when dealing with large disturbances mostly related to not well-known deterministic behaviors (e.g. load torque of a motor under incompletely specified operating conditions). The *set-membership* paradigm relying on unknown but bounded uncertainties (Schweppe, 1968) can lead to descriptions requiring no assumption about the probability distributions. Variants of ellipsoidal state sets have been proposed (Bertsekas & Rhodes, 1971; Kurzhanskiy & Valyi, 1997) as well as solutions addressing robustness issues (Petersen & Savkin, 1999). Interval analysis (Jaulin, Kieffer, Didrit, & Walter, 2001; Moore, 1966) has also led to state bounding algorithms, either based on set predictions/intersections (Chisci, Garulli, & Zappa, 1996; Combastel, 2003; Jaulin et al., 2001; Raïssi, Ramdani, & Candau, 2004) resembling Kalman filtering, or based on interval observers (Combastel, 2013; Gouzé, Rappaport, & Hadj-Sadok, 2000; Mazenc & Bernard, 2011; Raïssi, Efimov, & Zolghadri, 2012). The latter usually provide computationally efficient set-membership estimations with proven stability, but essentially deal with interval hulls which may be a rough enclosure of the consistent state sets. Though better taking domain shapes into account, single ellipsoids may hardly give tight enclosures of interval vectors and general polytopes (Ziegler, 1994) often suffer from the complexity of vertices/facets enumeration wrt the space dimension. This motivates the focus of this paper on zonotopes (Kühn, 1998; Ziegler, 1994), a class of polytopes

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whose shape is implicitly represented by a rectangular matrix. Basic operations over zonotopes often reducing to simple matrix calculations, zonotopic state bounding observers have already been proposed (Alamo, Bravo, & Camacho, 2005; Combastel, 2003, 2005; Le, Stoica, Alamo, Camacho, & Dumur, 2013; Montes de Oca, Puig, & Blesa, 2012). In Le et al. (2013), the so-called P-radius is introduced as a zonotope size criterion and permits to prove that zonotopic state estimates have non-increasing sizes in time, for time-invariant systems under the assumption that the required complexity reduction operator (Kühn, 1998) has a negligible influence.

By minimizing a matrix norm serving as a zonotope size criterion named F-radius (F_W-radius if the norm is weighted) and by using the covariation of a zonotope which is a set-membership analog to covariance, a Zonotopic Kalman Filter (ZKF) is here derived from a discrete time-varying model subject to unknown but bounded uncertainties. ZKF computes minimal zonotopic sets enclosing all the admissible states. Explicit links between the set-membership and the stochastic paradigms for Kalman filtering are given. A robust stability analysis using Linear Matrix Inequalities (LMI) and fully taking the reduction operator into account is proposed. Compared to previous zonotopic observers using a singular value decomposition (svd), the complexity order of ZKF is significantly improved.

The paper organization begins with preliminaries in Section 2 and the problem formulation in Section 3. The optimal observer gain minimizing the F_W-radius of the predicted zonotopic state set is studied in Section 4. The ZKF algorithm is given in Section 5. After comparison with the stochastic Kalman Filter in Section 6, the robust stability is analyzed in Section 7. A numerical example illustrates the efficiency of ZKF in Section 8.

2. Preliminaries

2.1. Matrix calculus (*Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Golub & Van Loan, 1996*)

$\partial_X f(X)$ is a short notation for $\partial f(X)/\partial X$. If the function f returns scalar values and $X = [X_{ij}]$ is a matrix, then $\partial_X f(X) = [\partial_{X_{ji}} f(X)]$. $tr(.)$ denoting the trace of a square matrix ($tr(A) = \sum_i A_{ii}$), and X, A, B, C being matrices of appropriate size, it comes:

$$tr(A) = tr(A^T), \qquad tr(AB) = tr(BA), \tag{1}$$

$$\partial_X tr(AX^T B) = A^T B^T, \tag{2}$$

$$\partial_X tr(AXBX^T C) = BX^T CA + B^T X^T A^T C^T. \tag{3}$$

The matrix M is spd if it is symmetric ($M = M^T$) and positive definite ($M \succ 0$ i.e. $\forall x \neq 0,\ x^T Mx > 0$). Then, M^{-1} exists and is spd. Similar definitions and properties hold for the negative case. The Schur complement of C in $L = [A, B; B^T, C]$ is $S = A - BC^{-1}B^T$ and $L \succ 0 \Leftrightarrow A \succ 0 \wedge S \succ 0$. Let $R = [r_1, \ldots, r_p] \in \mathbb{R}^{n\times p}$ be a matrix with p non zero columns and $M \in \mathbb{R}^{n\times n}$. For any such R, if M is spd, then $R^T MR$ is spd and $tr(R^T MR) > 0$. Similarly, $M \prec 0 \Rightarrow tr(R^T MR) < 0$.

Let $W \in \mathbb{R}^{n\times n}$ be a spd matrix and $R \in \mathbb{R}^{n\times p}$, $\|R\|_{F,W} = \sqrt{tr(R^T WR)}$ is the weighted Frobenius norm of R. $\|R\|_F$, obtained for $W = I_n$ (identity: no weight), is invariant under orthogonal transformations. Also: $\|R\|^2_{F,W} = \sum_{i=1}^{p} \|r_i\|^2_W$, where $\|r_i\|_W = \sqrt{r_i^T W r_i}$ is a weighted vector norm in $\mathbb{R}^n$.

2.2. Zonotopes

A zonotope $\langle c, R\rangle \subset \mathbb{R}^n$ with center $c \in \mathbb{R}^n$ and generator matrix $R \in \mathbb{R}^{n\times p}$ is a polytopic set defined as the linear image of the unit hypercube $[-1, +1]^p \subset \mathbb{R}^p$ by R:

$$\langle c, R\rangle = \{c + Rs,\ \|s\|_\infty \leq 1\}. \tag{4}$$

$\langle R\rangle = \langle 0, R\rangle$ is called centered zonotope. Any permutation of the columns of R leaves it invariant. The Minkowski sum of two sets S_1 and S_2 is $S_1 \oplus S_2 = \{s_1 + s_2,\ (s_1, s_2) \in S_1 \times S_2\}$. The linear image of the set $S \subset \mathbb{R}^n$ by $L \in \mathbb{R}^{q\times n}$ is $L \odot S = \{Ls,\ s \in S\}$. Zonotopes form a class of polytopic sets implicitly represented by matrices and leading to efficient set computations (Kühn, 1998; Ziegler, 1994). This class is closed under Minkowski sum $\oplus$ (computed from a matrix concatenation like $[R_1, R_2]$ in (5)) and linear image $\odot$ (computed from a matrix product like LR in (6)):

$$\langle c_1, R_1\rangle \oplus \langle c_2, R_2\rangle = \langle c_1 + c_2, [R_1, R_2]\rangle, \tag{5}$$

$$L \odot \langle c, R\rangle = \langle Lc, LR\rangle. \tag{6}$$

$$\langle c, R\rangle \subset \langle c, b(R)\rangle, \quad b(R) = diag(|R|\mathbf{1}), \tag{7}$$

(7) shows how a zonotope $\langle c, R\rangle$ can be enclosed within an aligned box (or interval hull) defined by $b(R) \in \mathbb{R}^{n\times n}$, where $|\cdot|$ is the element-by-element absolute value operator, $\mathbf{1}$ is a column vector of ones, and $diag(.)$ returns a diagonal matrix from a vector of diagonal elements. Such a box enclosure usually being too conservative, a reduction operator can be used: Kühn (1998). In this work, a weighted version of the reduction operator first introduced in Combastel (2003) and improved in Combastel (2005) will be denoted $\downarrow_{q,W}$, where $q \geq n$ specifies the maximum number of columns of matrix $\downarrow_{q,W} R$ satisfying the inclusion property $\langle R\rangle \subset \langle \downarrow_{q,W} R\rangle$. The reduction operator $\downarrow_{q,W}$ first consists in sorting the columns of $R \in \mathbb{R}^{n\times p}$ on decreasing weighted vector norm $\|\cdot\|_W$ (8) and enclosing the set $\langle R_<\rangle$ generated by the $p - q + n$ smaller columns into a box:

$$R = [r_1, \ldots, r_j, \ldots, r_p], \qquad \|r_j\|^2_W \geq \|r_{j+1}\|^2_W, \tag{8}$$

If $p \leq q$, Then $\downarrow_{q,W} R = R$, Else

$$\downarrow_{q,W} R = [R_>,\ b(R_<)] \in \mathbb{R}^{n\times q}, \tag{9}$$

$$R_> = [r_1, \ldots, r_{q-n}], \qquad R_< = [r_{q-n+1}, \ldots, r_p]. \tag{10}$$

3. Problem formulation

The zonotopic state observation of systems modeled by uncertain time-varying discrete-time dynamics as in (11)–(12) is the main subject of this paper:

$$x_{k+1} = A_k x_k + B_k u_k + E_k v_k, \quad v_k \in \langle 0, I_{n_v}\rangle, \tag{11}$$

$$y_k = C_k x_k + D_k u_k + F_k w_k, \quad w_k \in \langle 0, I_{n_w}\rangle, \tag{12}$$

$$x_0 \in \langle c_0, R_0\rangle \subset \mathbb{R}^{n_x}. \tag{13}$$

n_M (resp. p_M) denoting the number of lines (resp. columns) of any matrix M, $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $v_k \in \mathbb{R}^{n_v}$, $w_k \in \mathbb{R}^{n_w}$ respectively refer to the state, the known input, the known output (measurement), the state evolution uncertainty, and the measurement uncertainty vectors. $\forall k \geq 0$, $A_k, B_k, E_k, C_k, D_k, F_k$ denote non empty matrices with appropriately fixed size. All the related matrix sequences, e.g. $A_. = (A_0, \ldots, A_k, \ldots)$, $k \in \mathbb{N}$, are not necessarily known *a priori*, but each matrix with time index k is assumed to be known at time k, possibly expressed as a matrix function, e.g. $A(.) : \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_x\times n_x}$, of a known (measured) scheduling parameter vector $\alpha_k \in \mathbb{R}^{n_\alpha}$ so that $A_k = A(\alpha_k)$, $B_k = B(\alpha_k)$, etc. Notice that a robust detectability assumption further stated in Section 7 will be required to prove the convergence of the proposed set-based observer. The initial state x_0 is assumed to belong to a zonotope $\langle c_0, R_0\rangle$ specified by its center $c_0 \in \mathbb{R}^{n_x}$ and non empty generator matrix $R_0 \in \mathbb{R}^{n_x\times p_{R_0}}$ (13), (4). $\forall k \geq 0,\ v_k \in [-1, +1]^{n_v} = \langle 0, I_{n_v}\rangle$ where I_{n_v} is the identity matrix of size $n_v \times n_v$ i.e. v_k is bounded by a unit hypercube expressed as a centered zonotope in (11), and idem for w_k in (12). For the sake of simplified notations, the discrete time index k can be omitted and the index $+$ replaces $k + 1$ in the following.