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Node-consistent core for games played over event trees[☆]

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ABSTRACT

We consider a class of dynamic games played over an event tree, where the players cooperate to optimize their expected joint payoff. Assuming that the players adopt the core as the solution concept of the cooperative game, we devise a node-decomposition of the imputations in the core such that each player finds it individually rational at each node to stick to cooperation rather than switching to a noncooperative strategy. We illustrate our approach with an example of pollution control.

1. Introduction

We consider a class of stochastic discrete-time dynamic games, where the uncertainty is represented by an exogenously given event tree, that is, the random process is not influenced by the players' actions. This class of games, which involves flow (control) and stock (state) variables, is useful to model competition and cooperation between players interacting repeatedly overtime.[1] As an example, the set of players could be firms belonging to the same industry, where each firm invests in advertising (control variable) to increase its goodwill (state variable), with the consumer demand for each firm's brand being a function of all firms goodwill stocks and some random event (weather, state of the economy, etc.). This class of games was initially introduced in Haurie, Zaccour, and Smeers (1990) and Zaccour (1987) to study noncooperative equilibria in the European natural gas market, which involves four suppliers competing over a long-term planning horizon in nine markets described by stochastic demand laws. The solution concept was termed *S*-adapted equilibrium, where the *S* stands for *sample* of realizations of the random process (see Haurie, Krawczyk, & Zaccour, 2012 for details). Genc, Reynolds, and Sen (2007), Genc and Sen (2008), Pineau and Murto (2003) and Pineau, Rasata, and Zaccour (2011) used this formalism to model deregulated electricity markets and predict equilibrium investments in different generation technologies.

In all the references above, the mode of play is noncooperative. Recently, Reddy, Shevkoplyas, and Zaccour (2013) assumed that the players can cooperate and proposed a time-consistent Shapley value to share the benefits of cooperation among the participating players. A cooperative solution is time consistent if, at any intermediate instant of time, the cooperative payoff-to-go dominates the noncooperative payoff-to-go, for each player. Put differently, time consistency means that, at any intermediate instant of time, the cooperative payoff-to-go belongs to the same cooperative solution chosen by the players at the beginning of the game. There is a rich literature in differential games focusing on the design of time-consistent mechanisms to sustain cooperation overtime. The concept of time consistency and its implementation in cooperative differential games was initially proposed in Petrosjan (1977). For a comprehensive coverage of this literature, see the book by Yeung and Petrosjan (2005) and the survey by Zaccour (2008). The main contribution in Reddy et al. (2013) with respect to this literature

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[1] The main difference between this class of games and classical stochastic games is that here players cannot influence the transition between one decision node (or state) and another one.

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is in the determination of a *time-consistent* Shapley value for the class of dynamic games played over event tree. The authors show that in any subgame starting at any node in the event tree, each player is better off receiving his expected Shapley payment rather than his expected Nash-equilibrium payoff.

In this paper, we consider the case where the players adopt the core as a solution concept to the cooperative game. In particular, we show that it is possible to construct an imputation that belongs to the *node-consistent* core. We use the term node-consistent to emphasize that the payoff-dominance property akin to sustainability of cooperation holds for any node (and not only period) in the event tree.

The literature on time consistency of the core in stochastic dynamic games is sparse, with Predtetchinski (2007) and Xu and Veinott (2013) being the closest to what we are doing here. Before highliting the differences with these two papers, we note that Chander and Tulkens (1997), Filar and Petrosjan (2000), Germain, Toint, Tulkens, and de Zeeuw (2003) and recently Lehrer and Scarsini (2013) also studied time consistency of the core (not necessarily using the same wording) in deterministic dynamic games. In Filar and Petrosjan (2000) the characteristic function determining transferable utility game (TU-game) at each time period changes over time according to some dynamics, and the values of the characteristic function for the whole game are determined as the sum of the corresponding values of the stage characteristic functions. In Lehrer and Scarsini (2013), the payoff of a coalition depends on the history of allocations, and any coalition is allowed to deviate at any time. Interestingly, a deviation induces a structural change, i.e., that the deviating coalition becomes the grand coalition in a new dynamic game, and this leads to the authors to introduce the new solution concept (the intertemporal core), which insures stability against such deviations. As it will be made clear in the sequel, we also account for deviations when computing the values of the characteristic function, and devise a node-consistent payment scheme to deter such deviations. We postpone the discussion of the two related papers by Chander and Tulkens (1997) and Germain et al. (2003) after setting our model and results, as it will be then much easier to contrast their contributions to this one.

Predtetchinski (2007)[2] considered a class of discrete-time stochastic dynamic games where one non-transferable utility game (NTU-game) from a finite set can be realized in each time period. The transition from a state to another is described by a Markov process. The author introduces the solution of strong sequential core for stationary cooperative games, i.e., defines the set of utilities that are robust to deviation by any coalitions, and provide conditions for non-emptiness of this core. At least two main differences can be pointed out with the present paper. First, in Predtetchinski (2007) the values of the characteristic functions are given, whereas here they are obtained as Nash equilibrium outcomes. Second, as he is analyzing NTU-games, there is no issue of determining transfers to maintain time consistency. Our focus is precisely defining a decomposition over time of imputations (or transfers) from the core to obtain the node-consistent core for the whole game.

Xu and Veinott (2013) are also interested by time-consistency of the core in a setting where the players' profits depend on a (same) random variable. Here, cooperation means that players can share resources and information about this random variable. The value of the characteristic function for a coalition is given by the supremum of its conditional expected total payoff. Their stochastic process is similar to ours, that is, the realization of the random variable does not depend on players' actions and all players face the same randomness. However, and very importantly, non-coalition members do not influence the coalition's payoff, which means, using economic jargon, that they are dealing with games with no externalities. This greatly simplifies the computation of the characteristic function values, which are then given by the solution of optimization problems. In our case, the value of a coalition deviating from the grand coalition is given by solving an equilibrium problem, meaning that non-coalition members have a say in what this coalition can achieve alone. Xu and Veinott determine the sequential stochastic core elements by calculating a saddle point of a specific Lagrangian.

Finally, we wish to point out one additional non-trivial difference with the papers discussed above, namely, the modeling of the dynamics of the game. Typically in the above discussed papers, the dynamics are related to payoffs in the stage game (which can also be history dependent), and the focus is on the derivation of conditions guaranteeing non-emptiness of the core. Here, we use a control-theoretic formalism, that is, we distinguish between flow (control) and (stock) state variables, with the evolution of the latter being dependent on players' decisions, and not on outcomes. This implies that each subgame starts in a position that summarizes the history of past decisions and the stochastic process. As shown in the empirical applications (see the above cited references in energy markets), we believe that our formalism, i.e., stochastic multistage games, has an interesting and natural practical appeal in applications in, e.g., economics, engineering and management science.

The rest of the paper is organized as follows: Section 2 recalls the main ingredients of the class of games of interest, and Section 3 deals with the node consistency of the core. We provide an illustrative example in Section 4, and briefly conclude in Section 5.

2. Game over event tree

We recall in this section the main ingredients of the class of discrete-time dynamic games player over an event tree (see Haurie et al., 2012 for more details). Let $\mathcal{T} = \{0, 1, \ldots, T\}$ be the set of periods. The exogenous stochastic process is represented by an event tree, which has a root node n^0 in period 0 and a set of nodes $\mathcal{N}^t = \{n_1^t, \ldots, n_{N_t}^t\}$ in period $t = 1, \ldots, T$. Each node $n_l^t \in \mathcal{N}^t$ represents a possible sample value of the history of the stochastic process up to time t. The tree graph structure represents the nesting of information as one time period succeeds the other. Denote by $a(n_l^t) \in \mathcal{N}^{t-1}$ the unique predecessor of node $n_l^t \in \mathcal{N}^t$ on the event-tree graph for $t = 1, \ldots, T$, and by $\mathcal{S}(n_l^t) \subset \mathcal{N}^{t+1}$ the set of all possible direct successors of node $n_l^t \in \mathcal{N}^t$ for $t = 0, \ldots, T-1$. A path from the root node n^0 to a terminal node n_l^T is called a *scenario*. Each scenario has a probability and the probabilities of all scenarios sum up to 1. We denote by $\pi(n_l^t)$ the probability of passing through node n_l^t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi(n^0) = 1$ and $\pi(n_l^T)$ is equal to the probability of the single scenario that terminates in (leaf) node n_l^T.

Denote by $M = \{1, \ldots, m\}$ the set of players. For each player $j \in M$, we define a set of decision variables indexed over the set of nodes. Denote by $u_j(n_l^t) \in \mathbb{R}^{m_j}$ the decision variables of player j at node n_l^t, and let $\underline{u}(n_l^t) = \left(u_1(n_l^t), \ldots, u_m(n_l^t)\right)$. Let $X \subset \mathbb{R}^p$, with p a given positive integer, be a state set. For each node $n_l^t \in \mathcal{N}^t$, $t = 0, 1, \ldots, T$, let $U_j^{n_l^t} \subset \mathbb{R}^{\mu_j^{n_l^t}}$, with $\mu_j^{n_l^t}$ a given positive integer, be the control set of player j. Denote by $\underline{U}^{n_l^t} = U_1^{n_l^t} \times \cdots \times U_j^{n_l^t} \times \cdots \times U_m^{n_l^t}$ the product control sets. A transition function $f^{n_l^t}(\cdot, \cdot) : X \times \underline{U}^{n_l^t} \mapsto X$ is associated with each node n_l^t. The state equations are given by

$$x(n_l^t) = f^{a(n_l^t)}\left(x\left(a\left(n_l^t\right)\right), \underline{u}\left(a\left(n_l^t\right)\right)\right), \tag{1}$$

$$\underline{u}\left(a\left(n_l^t\right)\right) \in \underline{U}^{a(n_l^t)}, \quad n_l^t \in \mathcal{N}^t,\ t = 1, \ldots, T. \tag{2}$$

[2] See also Predtetchinski, Herings, and Perea (2006); Predtetchinski, Herings, and Peters (2002).