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Brief paper

Stability analysis of linear delay systems with cone invariance ☆

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ABSTRACT

This paper is concerned with the stability and input–output gain analysis of linear delay systems with cone invariance. Based on the partial ordering over a cone, the monotonicity of the trajectory of the cone-preserving systems with constant delays is first studied. Then, by comparing the trajectory of the constant delay systems and that of time-varying delay systems, we prove that a cone-preserving system with interval time-varying delays is asymptotically stable if and only if the corresponding delay-free system is asymptotically stable. This implies that the stability of a cone-preserving system is insensitive to the magnitude of the delays. Moreover, based on the cone-induced norms, an explicit characterization on the cone-induced gain of an input–output cone-preserving system is given in terms of system matrices. Finally, numerical examples are provided to illustrate the theoretical results.

1. Introduction

Internally positive systems, whose state and output are constrained in the nonnegative orthant whenever the initial condition and input are nonnegative, have drawn tremendous attention in the past decade. This is partly due to their broad applications in systems biology (de Jong, 2002), ecology and power control of wireless networks (Zappavigna, Charalambous, & Knorn, 2012) and so on. Both behavioral analysis and synthesis of positive dynamic systems are available in the literature, and one can refer to Ait Rami (2011), Ait Rami and Tadeo (2007), Colaneri, Middleton, Chen, Caporale, and Blanchini (2014), Kaczorek (2002, 2011), Li, Lam, and Shu (2010) Li, Lam, Wang, and Date (2011), Liu and Dang (2011), Wang and Huang (2013), Zhao, Liu, Yin, and Li (2014), Zhao, Zhang, Shi, and Liu (2012) and the references therein. As a natural generalization of internally positive systems, we consider a special class of systems with a proper cone being an invariant set. Such systems, also referred to as cone-preserving systems and monotone control systems (Angeli & Sontag, 2003), have applications in rendezvous of multi-agent systems. The rendezvous problem (Bhattacharya, Tiwari, Fung, & Murray, 2009; Tiwari, Fung, Bhattacharya, & Murray, 2004) of multiple agents (for instance, can be vehicles, sensors) subject to communication delays can be formulated as a linear delay system with cone invariance (with some restrictions that the initial conditions must be given in an appropriately defined polyhedral cone). A complete characterization on the polyhedra invariance of a continuous-time linear system is given in Castelan and Hennet (1993). In this paper, instead of studying dynamic systems leaving a specific cone invariant, we focus on the asymptotic stability and input–output gain analysis for a linear delay system which is invariant on a general proper cone.

For linear systems under nonnegativity constraint, many delay related robust properties have been reported. For instance, it was pointed out in Haddad and Chellaboina (2004) that a positive system with constant delays is asymptotically stable if and only if the sum of all the system parameter matrices is Hurwitz. In Liu, Yu, and Wang (2010) and Ngoc (2013), it was reported that both asymptotic stability and exponential stability of positive systems with bounded time-varying delays can be achieved no matter how large the delays are. Recently, for cone-preserving systems with constant delays, it was proved in Tanaka, Langbort, and Ugrinovskii (2013) that the magnitude of the delays does not affect their asymptotic stability by using the fact that the spectral radius of the cone-preserving transfer function along the imaginary axis attains its maximum at frequency zero. For the input–output gain analysis, both L_1 and L_∞ gains are often employed as performance measures for positive systems (Briat, 2013; Chen, Lam, Li, & Shu, 2013). A delay-independent characterization of the weighted

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L_1-gain for positive systems with constant delays was presented in Haddad, Chellaboina, and Rajpurohit (2004). In Briat (2013), the characterization of the L_∞-gain of a positive system was derived from the L_1-gain computation of its dual system. Different from Briat (2013), we emphasize that the ∞-norm can also be regarded as a norm induced by the partial ordering defined with respect to the nonnegative orthant. This viewpoint enables us to establish explicit characterizations on the cone-induced gains for input–output cone-preserving systems.

The delay robustness in positive systems naturally gives rise to a question: is this delay-independent property due to the cone invariance of the system rather than nonnegativity? In this paper, we provide a positive answer to this question, that is, we aim at proving that the asymptotic stability of a cone-preserving system with bounded time-varying delays is insensitive to the magnitude of the delays. By studying the partial ordering of the set of state trajectories, the monotonicity of state trajectories with respect to a given cone is analyzed. Along this line, based on the cone-induced norms, we further explicitly characterize the cone-induced gain of cone-preserving systems in terms of system matrices. Note that the linear co-positive Lyapunov–Krasovskii functional method in Haddad and Chellaboina (2004) is based on the L_1-norm, which is not suitable for analyzing systems defined on a general proper cone. In this connection, we will adopt cone-induced norms in this paper, which facilitates the analysis of state trajectories over cones. This viewpoint serves as a basis for both stability and input–output gain analysis. It is worth mentioning that the techniques utilized in Liu et al. (2010) have to resort to the explicit solutions of the system and thus they are not easily applicable for the stability analysis of cone-invariant systems. The stability analysis of cone-preserving systems with constant delays in Tanaka et al. (2013) was performed from an input–output viewpoint and relies on the transfer function of a linear time-invariant system, therefore it is no longer valid when time-varying delays are encountered.

The main technical novelty and significance of the work in this paper lie in that the developed approach is based on the monotonicity of the state trajectory of the constant delay system (with special initial conditions) and its comparison with the time-varying delay system, instead of relying on explicit expressions of the state trajectory. Therefore, this method is much straightforward for the analysis of cone-preserving systems. The general idea of our proof is based on the comparison principle. More specifically, our proof can be divided into the following steps. We first prove that for a constant delay system with cone invariance under a special initial condition (which is selected based on Lemma 3), its state trajectory is monotonic with respect to the partial order induced by the cone. This also ensures the attractivity of the constant delay system. Then, the time-varying delay system is analyzed by comparison with the constant delay system. This paper can be viewed as important extensions of some recent works concerning the stability analysis of positive delay systems (Liu et al., 2010; Ngoc, 2013) as well as L_∞-gain analysis for positive systems (Briat, 2013). The results of this paper may also have potential applications in the analysis of positive descriptor system (Ait Rami & Napp, 2012, 2014) since the admissible initial conditions of a positive descriptor system can be represented by a conic set.

2. Preliminaries

In this section, some basic notations and lemmas in book Berman and Plemmons (1994) will be recalled. $\mathbb{R}^n$ and $\mathbb{R}^n_+$ denote the set of all real vectors and all vectors with nonnegative entries, respectively. $\mathbf{1}_n$ denotes an n-dimensional vector with each entry equal to 1. The set of all vector-valued continuous functions defined on $[-\tau, 0]$ is denoted by $\mathbb{C}([-\tau, 0], \mathbb{R}^n)$. The boundary of a set S and its interior are denoted by ∂S and $\operatorname{int} S$, respectively. Given a set S, S^G denotes the set consisting of all finite nonnegative linear combinations of the elements of S. S^*, the dual of S, is defined by $S^* = \{z \in \mathbb{R}^n : (z, y) \geq 0 \text{ for all } y \in S\}$, where $(\cdot, \cdot)$ stands for the inner product. For a set $S \subseteq \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{m\times n}$, by AS we mean that $AS = \{Ax : x \in S\}$. A set is defined to be a cone if $K = K^G$. A convex cone is pointed if $K \cap (-K) = \{0\}$ and is solid if $\operatorname{int} K \neq \emptyset$. A closed, pointed, solid convex cone K is called a proper cone. Throughout this paper, we always assume that the given cone is proper. A proper cone K induces a partial ordering in $\mathbb{R}^n$ via $y \preceq_K x$ if and only if $x - y \in K$.

Given a proper cone $K \subset \mathbb{R}^n$, a matrix $A \in \mathbb{R}^{m\times n}$ is called K-nonnegative if $AK \subseteq K$ and is called K-positive if $A(K\backslash\{0\}) \subseteq \operatorname{int} K$. A square matrix A is called cross-positive on K if for all $y \in K, z \in K^*$ such that $(z, y) = 0$, we have $(z, Ay) \geq 0$. Note that the sum of a cross-positive matrix and a K-nonnegative matrix is still cross-positive on K. A square matrix is called Metzler if all its off-diagonal entries are nonnegative. For any matrix $A \in \mathbb{R}^{n\times n}$, $\mu(A) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ denotes the spectral abscissa of A, where $\sigma(A)$ is the spectrum of A.

The next lemma directly follows from Schneider and Vidyasagar (1970, Lemma 6 and Theorem 2).

Lemma 1 (*Schneider & Vidyasagar, 1970*)**.** *Let $K \subset \mathbb{R}^n$ be a proper cone and let A be cross-positive on K. Then there exist a sequence of matrices A_i and a sequence of real numbers $\alpha_i \geq 0$ satisfying that $(A_i + \alpha_i I)(K\backslash\{0\}) \subseteq \operatorname{int} K$ and $\lim_{i\to\infty} A_i = A$.*

In the following we introduce the cone-induced vector norm and the cone-induced matrix norm, which will be needed in the sequel.

Definition 1 (*Berman & Plemmons, 1994, pp. 5–6*)**.** Let $K \subset \mathbb{R}^n$ be a proper cone and let $v \in \operatorname{int} K$. Then an order interval B_v is defined as $B_v = \{x \in \mathbb{R}^n : -v \preceq_K x \preceq_K v\}$. The set B_v defines a norm on $\mathbb{R}^n$:

$$\|x\|_v = \inf\{t \geq 0 : x \in tB_v\}.$$

Note that the cone-induced vector norm is monotonic with respect to cone K, that is, $0 \preceq_K x \preceq_K y$ implies that $\|x\|_v \leq \|y\|_v$. The cone-induced matrix norm of a square matrix A is defined as $\|A\|_v = \sup_{\|x\|_v=1} \|Ax\|_v$ and it satisfies that $\|A\|_v = \|Av\|_v$ provided that A is K-nonnegative.

3. Stability analysis

In this section, we shall point out a consequence of the cone invariance, that is, the asymptotic stability is robust against bounded time-varying delays. In what follows, we first give a characterization on the cone invariance of continuous-time linear systems.

Lemma 2. *Given proper cones $K_x \subset \mathbb{R}^n$ and $K_u \in \mathbb{R}^m$, suppose that A is cross-positive on K_x, A_d is K_x-nonnegative and $BK_u \subseteq K_x$. Then, for any initial condition $x(s) = \phi(s) \in K_x$ $(s \in [-d, 0])$, any input $u(t) \in K_u$ $(t \geq 0)$ and any delays $d(t)$ satisfying $0 \leq d(t) \leq d$, the state trajectory of system*

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + Bu(t) \tag{1}$$

satisfies that $x(t) \in K_x$ for all $t \geq 0$.

Proof. Our goal is to prove that for any given $T > 0$ and $\phi(s) \in \operatorname{int} K_x$ $(s \in [-d, 0])$, $x(t) \in K_x$ holds for all $t \in [0, T]$. Then, by the continuous dependence of the solution of system (1) on the initial value, together with the fact that cone K_x is closed, one can deduce that for any $\phi(s) \in K_x$ $(s \in [-d, 0])$, $x(t) \in K_x$ holds for all $t \in [0, T]$. Note that for any cross-positive matrix A, according to Lemma 1, one can always find a sequence of matrices A_i and a