Automatica 53 (2015) 85–93

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stability and performance analysis of saturated systems via partitioning of the virtual input space ☆

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ARTICLE INFO

Article history:
Received 2 May 2014
Received in revised form
7 August 2014
Accepted 3 December 2014

Keywords:
Actuator saturation
Constrained control
Domain of attraction
Input saturation
Nonlinear $\mathcal{L}_2$ gain
Piecewise quadratic Lyapunov functions
Stability analysis

ABSTRACT

This paper revisits the problem of estimating the domain of attraction and the nonlinear $\mathcal{L}_2$ gain for systems with saturation nonlinearities. We construct a virtual input space from the algebraic loop contained in systems, and divide this virtual input space into several regions. In one of these regions, none of the virtual inputs saturate. In each of the remaining regions, there is a unique virtual input that saturates everywhere with the time-derivative of its saturated signal being zero. These special properties of the virtual inputs in different regions of the virtual input space are combined with an existing piecewise quadratic Lyapunov function that contains the information of virtual input saturation to arrive at a set of less conservative stability and performance conditions, from which we can obtain a larger level set of the piecewise quadratic Lyapunov function as an estimate of the domain of attraction and a tighter scalar function of the bound on the $\mathcal{L}_2$ norm of the exogenous input as an estimate of the local nonlinear $\mathcal{L}_2$ gain. Simulation results indicate that the proposed approach has the ability to obtain a significantly larger estimate of the domain of attraction and a significantly tighter estimate of the nonlinear $\mathcal{L}_2$ gain than the existing methods.

1. Introduction

Over the past decades, systems with saturation nonlinearities have been extensively studied in the research community, due in part to their ubiquity in engineering and in part to the theoretical challenges they pose in control theory. Enormous amount of research has been devoted to the analysis and synthesis of such systems. For example, the problems of global controllability and global stabilization have been studied in depth in Sussmann, Sontag, and Yang (1994), and Teel (1992), and semi-global stabilization by linear feedback has been studied in Lin (1998), Lin and Saberi (1993) and Saberi, Lin, and Teel (1996).

The estimations of the domain of attraction (Alamo, Cepeda, & Limon, 2005; Cao & Lin, 2003; Dai, Hu, Teel, & Zaccarian, 2009; Gomes da Silva & Tarbouriech, 2005; Hu & Lin, 2001, 2003; Hu, Teel, & Zaccarian, 2006; Li & Lin, 2013) and the nonlinear $\mathcal{L}_2$ gain (Crawshaw & Vinnicombe, 2000; Hu et al., 2006; Hu, Teel, & Zaccarian, 2008; Wu & Soto, 2004; Zaccarian & Teel, 2004) play a very important role in analysis of linear systems under saturated feedback. This is especially so for systems that cannot be globally or semi-globally stabilized. Level sets of Lyapunov functions and scalar functions of the bound on the $\mathcal{L}_2$ norm of the exogenous input that characterize the disturbance rejection are commonly used as estimates of the domain of attraction and the local nonlinear $\mathcal{L}_2$ gain, respectively, for such systems. Key to obtaining such estimates of less conservatism is the treatment of the saturation function and the construction of the Lyapunov function. Much effort has been made in improving both of these two aspects.

In treating the saturation function, a popular way is to bound it with a global or regional sectors (see, for example, Gomes da Silva & Tarbouriech, 2005, Hu et al., 2006). To reduce the conservatism, a convex hull representation of saturation functions was proposed in Hu and Lin (2001), which results in a linear differential inclusion

☆ This work was supported in part by the National Natural Science Foundation of China under Grant Nos. 60221003 and 61273105. The material in this paper was partially presented at the 33rd Chinese Control Conference, July 28–30, 2014, Nanjing, China. This paper was recommended for publication in revised form by Associate Editor Emilia Fridman under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2014.12.033
0005-1098/

for saturated systems. Moreover, the authors of Alamo et al. (2005) presented an improved treatment with more auxiliary matrices than the representation in Hu and Lin (2001). A generalized sector condition that is able to provide a tighter bound of the saturation function than the linear sector was proposed in Hu, Huang, and Lin (2004). Sector-like conditions that characterize the relationship between saturation/deadzone functions and their time-derivative functions were derived in Dai et al. (2009).

With their simplicity, the quadratic Lyapunov functions are most commonly used Lyapunov functions in stability analysis of nonlinear systems. Many attempts have been made to reduce the conservatism associated with the quadratic Lyapunov functions. For example, the polynomial Lyapunov functions were adopted for polynomial systems (Chesi, 2004) and the maximum or minimum of a group of quadratic Lyapunov functions was studied for the time-delay systems (Xie, Shishkin, & Fu, 1997). Furthermore, a multiple Lyapunov function without requirements of its continuity and monotonousness can be found in Ahmadi and Parrilo (2008) and Branicky (1998).

For linear systems under a saturated linear feedback, some non-quadratic Lyapunov functions have been well investigated for both stability analysis and controller design. A composite quadratic Lyapunov function is composed from a group of quadratic functions in Hu and Lin (2003). A level set of this composite Lyapunov function is the convex hull of the corresponding level sets of the individual quadratic Lyapunov functions. A saturation-dependent Lyapunov function was proposed in Cao and Lin (2003) that takes into account the severity of the actuator saturation. An integral of the saturation/deadzone function was added to a quadratic Lyapunov function to form a Lure–Postnikov type Lyapunov function (Gomes da Silva, Tarbouriech, & Reginatto, 2002; Kapila, Sparks, & Pan, 2001). This Lure–Postnikov type Lyapunov function was generalized in Dai et al. (2009) to a piecewise quadratic Lyapunov function of an augmented state vector that contains the system states and the saturation/deadzone function. All these Lyapunov functions are generalized from quadratic Lyapunov functions and lead to substantial improvement in the estimations of the domain of attraction and the nonlinear $\mathcal{L}_2$ gain.

The introduction of the saturation/deadzone function in the Lure–Postnikov type Lyapunov function (Gomes da Silva et al., 2002; Kapila et al., 2001) and the piecewise quadratic Lyapunov function (Dai et al., 2009) provides an extra degree of freedom in the resulting stability and performance conditions. However, the information inherent in the saturation/deadzone function has not been fully exploited and entails further exploration. In this paper, we will present the idea of partitioning the virtual input space constructed from the algebraic loop of saturated systems. In particular, we will divide an m-dimensional virtual input space into $m+1$ regions. In one of these regions, none of the virtual inputs saturate. In each of the remaining regions, exactly one virtual input saturates everywhere with the time-derivative of its saturated signal being zero, and none of the remaining virtual inputs stay saturated everywhere in the region.

In Dai et al. (2009), the negative definiteness of a certain matrix ensures global/regional stability of the system. In the regions of the virtual input space where the time-derivative of the saturated signal of the kth virtual input is zero, we only need to guarantee the negative definiteness of the matrix in Dai et al. (2009), with the row and column corresponding to the kth virtual input removed, as the term in the derivative of the piecewise quadratic Lyapunov function corresponding to the saturated signal of the kth virtual input no longer exists. This will reduce conservatism in the stability and performance conditions and hence lead to a larger estimate of the domain of attraction and a tighter estimate of the nonlinear $\mathcal{L}_2$ gain for saturated systems than the method proposed in Dai et al. (2009).

The remainder of this paper is organized as follows. In Section 2, we review some treatments of the saturation function and introduce a partitioning of the virtual input space. In these partitions the properties of saturation functions are explored. The problem statement is also made in this section. In Section 3, based on these properties and the piecewise quadratic Lyapunov function (Dai et al., 2009), we establish the stability and performance conditions for the estimations of the domain of attraction and the nonlinear $\mathcal{L}_2$ gain. The optimization problems are formulated to maximize/minimize such estimates. Section 4 provides some simulation results to illustrate the effectiveness of the results in Section 3. Section 5 concludes the paper.

Notation. For $u \in \mathcal{L}_2$, $\|u\|_2 = \left(\int_0^\infty u^{\mathrm{T}}(t)u(t)dt\right)^{\frac{1}{2}}$. For a square matrix A, $\mathrm{He}(A) := A + A^{\mathrm{T}}$. For two integers l_1 and $l_2 \geq l_1$, $I[l_1, l_2]$ denotes the set of integers $\{l_1, l_1+1, \ldots, l_2\}$. For an integer m, let $\mathcal{K}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathcal{K}$. Suppose that these elements of $\mathcal{K}$ are labeled as K_i, $i \in I[1, 2^m]$. Let $K_i^- = I - K_i$. Clearly, $K_i^- \in \mathcal{K}$. Let I_m denote the identity matrix of dimension m, and $0_{n\times m}$ the $n \times m$ zero matrix.

2. Preliminaries and problem statement

2.1. Treatments of the saturation function

Consider a system with saturation described in the following form:

$$\begin{cases} \dot{x} = Ax + B_y \mathrm{sat}(y) + B_\omega \omega, \\ y = C_y x + D_{yy}\mathrm{sat}(y) + D_{y\omega}\omega, \\ z = C_z x + D_{zy}\mathrm{sat}(y) + D_{z\omega}\omega, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $y \in \mathbf{R}^m$ contains all the variables affected by saturation/deadzone, $\omega \in \mathbf{R}^r$ is the exogenous input such as the reference and disturbances, $z \in \mathbf{R}^p$ is the performance output, and sat : $\mathbf{R}^m \to \mathbf{R}^m$ denotes the saturation function, which is defined as $\mathrm{sat}(y) = [\mathrm{sat}(y_1), \mathrm{sat}(y_2), \ldots, \mathrm{sat}(y_m)]^{\mathrm{T}}$, $\mathrm{sat}(y_j) = \mathrm{sgn}(y_j)\min\{1, |y_j|\}$. Many linear systems with saturation components, such as anti-windup systems, can be transformed into the form of system (1). By the relationship $\mathrm{dz}(u) = u - \mathrm{sat}(u)$, where $\mathrm{dz}(\cdot)$ denotes the deadzone function, system (1) can be equivalently converted into a system with deadzones, which is considered in Dai et al. (2009) and Hu et al. (2006). When $D_{yy} = 0$, system (1) reduces to a linear system with saturated linear feedback. When $D_{yy} \neq 0$, system (1) contains the following algebraic loop,

$$y = C_y x + D_{yy}\mathrm{sat}(y) + D_{y\omega}\omega. \tag{2}$$

This algebraic loop is said to be well-posed if there exists a unique solution y for each $C_y x + D_{y\omega}\omega$. A necessary and sufficient condition for the well-posedness is that the values of $\det(I_m + (I_m - D_{yy})^{-1}D_{yy}K_i)$, $i \in I[1, 2^m]$, are all nonzero and have the same sign. One can easily verify this condition by Claim 2 in Hu et al. (2006). Throughout this paper, the well-posedness of the algebraic loop (2) is assumed to be satisfied.

Next, we review some treatments of the saturation functions as found in Dai et al. (2009), Gomes da Silva and Tarbouriech (2005) and Hu and Lin (2001), which will be used for the stability and performance analysis for system (1) in the next section.

Lemma 1. *Given* $v = [v_1\ \ v_2\ \ldots\ v_m]^{\mathrm{T}} \in \mathbf{R}^m$ *such that* $|v_j| \leq 1,\ \forall\ j \in I[1, m]$, *the following inequality holds for any diagonal matrix* $S \in \mathbf{R}^{m\times m}$ *satisfying* $S > 0$,

$$(u - \mathrm{sat}(u))^{\mathrm{T}} S\,(\mathrm{sat}(u) - v) \geq 0, \quad \forall\, u \in \mathbf{R}^m.$$