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Brief paper

Open-loop Nash equilibrium in polynomial differential games via state-dependent Riccati equation ☆

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ABSTRACT

This paper studies finite- as well as infinite-time horizon nonzero-sum polynomial differential games. In both cases, we explore the so-called state-dependent Riccati equations to find a set of strategies that guarantee an open-loop Nash equilibrium for this particular class of nonlinear games. Such a method presents advantages in simplicity of the design of equilibrium strategies and yields computationally effective solution algorithms. We demonstrate that this solution leads the game to an ε- or quasi-equilibrium- and provide an upper bound for this ε quantity. The proposed solution is given as a set of N coupled polynomial Riccati-like state-dependent differential equations, where each equation includes a p-linear form tensor representation for its polynomial part. We provide an algorithm for finding the solution of the state-dependent algebraic equation in the infinite-time case based on a Hamiltonian approach and give conditions on the existence of such stabilizing solutions for a third order polynomial. A numerical example is presented to illustrate effectiveness of the approach.

1. Introduction

Among many different fields such as engineering, ecology, management and economics, we find situations that involve several Decision-Makers (or Players) with different goals or objectives interlinked by the same decision process. This means that all of them are interacting and influencing each other by the decision they made: the action taken by any of the participants affects the others, and vice versa. Particularly, when the evolution of the underlying decision process evolves in time, this type of problems are often optimized using the Theory of Dynamic Games (Cruz & Xiaohuan, 2009; Dockner, Jorgensen, van Long, & Sorger, 2000; Gu, 2008; Huang, Caines, & Malhamé, 2007; Jorgensen, Martin-Herran, & Zaccour, 2010; Jungers, Castelan, De Pieri, & Abou-Kandil, 2008; Ma & Peng, 1999; Ma, Wang, Bo, & Guo, 2011; Maler & de Zeeuw, 1998; Maler, Xepapadeas, & de Zeeuw, 2003; Mukaidani, 2013; Nourian & Caines, 2013; Semsar-Kazerooni & Khorasani, 2009; Wang, Huang, & Unbehauen, 1999; Zhang, Cui, & Luo, 2013). This theory was initiated in the works of Isaacs (1965): he focused mainly on zero-sum games. Later on, the nonzero-sum differential games were introduced in Ho (1970) and Starr and Ho (1969a,b). In such games, each player looks for minimization of his own individual criterion. The paper (Starr & Ho, 1969b) derived sufficient conditions of existence of a linear feedback equilibrium for a finite planning horizon, but only in the case of linear–quadratic games governed by linear dynamics and quadratic criterion. (See Engwerda, 2005 for a detailed survey and Engwerda, 1998, 2000 and Engwerda, van Aarle, & Plasmans, 1999; Engwerda, van de Broek, & Schumacher, 2000). Usually, to resolve a conflict situation, the Nash-equilibrium (or, in general case, ε-Nash equilibrium) is applied (Basar & Olsder, 1999; Jimenez & Poznyak, 2006; Nash, 1950; Tanaka & Yokoyama, 1991). It is recognized that Nash equilibrium is a natural solution in a noncooperative context. However, if we deal with complex nonlinear dynamics, it seems more appropriate to apply the concept of ε-Nash equilibrium, since it allows one more flexibility in the selection of equilibrium strategies (see Basar & Olsder, 1999, Tanaka & Yokoyama, 1991).

In many applications, the originally linear modeling cannot fit all the situations in practice, which are mostly nonlinear by nature. Therefore, we need to extend the equilibrium concepts to a certain

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class of nonlinear systems, namely, polynomial systems. Polynomial dynamics represents an important class of nonlinear dynamical systems, since it can approximate a large variety of intrinsically nonlinear functions, keeping the complexity on a manageable and pre-specified level. Compared to the linear–quadratic case, there are not so many works on nonlinear differential games and, particularly, to the best of authors' knowledge, no results have been obtained for polynomial differential games (excluding a short conference version of this paper Jimenez-Lizarraga, Basin, Rodriguez, & Rodriguez, 2013). A few recent papers related to nonlinear differential games can be mentioned. The paper (Kossiorisa, Plexousakis, Xepapadeas, de Zeeuwe, & Maler, 2008) presents a solution in a particular case of a nonlinear game representing a pollution and resource management problem. The paper (Sorger, 1995) identifies the potentially chaotic behavior in a Markovian Nash equilibrium in a duopoly discrete-time model of advertising competition. The recent paper (Zhang, Wei, & Liu, 2011) proposes an iterative adaptive dynamic programming method to solve a particular type of games called two players zero-sum games. Another important reference close to this work is (Zhang & Feng, 2008), where a two players nonzero sum Nash game is applied to solve the problem of H_2/H_∞ control. All these publications express the interest in finding equilibrium strategies in complex non-linear systems.

In this paper, we develop the State-Dependent Riccati Equations (SDRE) approach (Barabanov, 1997; Barabanov, Ortega, & Escobar, 2008; Basin & Calderon-Alvarez, 2009; Basin, Perez, & Skliar, 2006; Çimen, 2008; Mracek & Cloutier, 1998) for a nonlinear polynomial game and derive a set of controls that leads to an open-loop ε-Nash equilibrium. For one player optimization problem (optimal control), the SDRE method has been proven to work well in many particular situations, providing a simple procedure for designing feedback controls (see Basin & Calderon-Alvarez, 2009, Basin et al., 2006, Çimen, 2008, Dong, Wang, & Gao, 2010, Gao, Shi, & Wang, 2007, Mracek & Cloutier, 1998, Rodrigues, 2004, 2007); however, the general case solution is quasi-optimal, that is, the SDRE approach leads only to an approximate result. This is why the SDRE method provides an ε-equilibrium for a game problem. Nevertheless, fast convergence of the obtained solution to the optimal one, a feedback form for the equilibrium controls, and numerical feasibility make the SDRE approach a valuable method. Both finite- and infinite-time cases are studied and solved for nonlinear polynomial games. In the finite-time case, the solution is represented as a set of N coupled state-dependent Riccati equations and a numerical procedure is suggested to find it. The infinite-time game problem is solved extending geometric methods developed for linear systems in Engwerda (2005). Note that in case of a scalar linear two players game, the associated characteristic matrix always has one stable eigenvalue (see Abou-Kandil & Bertrand, 1986); however, there is no similar result, to the best of the author's knowledge, for nonlinear polynomial games. The dependence of the characteristic matrix on x in the polynomial case makes the linear results inapplicable. Thus, the conditions on the parameters of a polynomial game presented in this paper determine existence or nonexistence of stabilizing solutions for a third order polynomial two players game. Moreover, it is established that the given controls stabilize the nonlinear game to the only equilibrium point: the origin.

For ε-Nash equilibrium provided by the obtained SDRE, the paper presents an explicit formula to find the upper bound for its possible deviation from the pure equilibrium ($\varepsilon = 0$). In the given example, those upper limits are explicitly calculated and shown to be less than 1% for each cost function. This indicates a close approximation of the optimal solution by an SDRE-based one and reveals quite a small level of degradation.

2. Problem statement

Consider the following polynomial differential game, where the players' dynamics is governed by the differential equation:

$$\dot{x}(t) = f(t, x) + \sum_{j=1}^{N} B^j(t)u^j(t) + d(t), \qquad x(t_0) = x_0, \tag{1}$$

and a *quadratic cost* functional as the individual performance index for each player:

$$L_T^i(u^i, u^{-i}) = \frac{1}{2}x^T(T)Q_f^i x(T) + \frac{1}{2}\int_{t_0}^{T}\left(x^T(t)Q_1^i(t)x(t) + \sum_{j=1}^{N} u^{jT}(t)R^{ij}(t)u^j(t)\right)dt, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the game, u^j is the control (action) of each j-player, which varies within a given region $U^j \subset \mathbb{R}^{m_j}$, j denotes the number of players $(j = 1, \dots, N)$, $B^j(t) \in \mathbb{R}^{n\times m_j}$ are the control matrices, and $d(t) \in \mathbb{R}^n$ is a continuous known exciting signal. The performance index $L_T^i(u^i, u^{-i})$ is given in the Bolza form, where u^i is the control of the i player and u^{-i} are the controls for the rest of the players ($-i$ is the counter-coalition collection of players counteracting to the player with index i). For each player, the purpose of the game is to achieve the minimization of his own performance index by selecting appropriate inputs. We also assume that:

$$Q^i(t) = Q^{i\top}(t) \geq 0, \qquad Q_f^i = Q_f^{i\top} \geq 0, \qquad R^{ii}(t) = R^{ii\top}(t) > 0, \qquad R^{ij}(t) = R^{ij\top}(t) \geq 0, \quad i \neq j. \tag{3}$$

We consider the nonlinear function $f(t, x)$ as a polynomial of n variables, components of the state vectors $x(t) \in \mathbb{R}^n$; this requires a special definition of the polynomial for degrees $n > 1$. Following the previous work (see Basin et al., 2006), a p-degree polynomial of a vector $x(t) \in \mathbb{R}^n$ is regarded as a p-linear form of n components of $x(t)$, that is to say:

$$f(t, x) = a_0(t) + a_1(t)x + a_2(t)x * x^{\mathsf{T}} + \cdots + a_s(t)x * \cdots s\ times \cdots * x. \tag{4}$$

Here, the involved parameters are: a_0 is a vector of dimension n, a_1 is a matrix of dimension $n \times n$, a_2 is a 3D tensor of dimension $n \times n \times n$, and a_s is an $(s+1)$D tensor of dimension $n \times \cdots (s+1)\ times \cdots \times n$, and $x * \cdots s\ times \cdots * x$ is a pD tensor of dimension $n \times \cdots s\ times \cdots \times n$, obtained by p times spatial multiplication of the vector x by itself. It is also possible to represent such a polynomial in a summation form:

$$f_k(t, x) = a_{0,k}(t) + \sum_{i=1}^{n} a_{1,k,i}(t)x_i + \sum_{i,j=1}^{n} a_{2,k,i,j}(t)x_i x_j + \cdots + \sum_{i_1,\dots,i_s=1}^{n} a_{1,k,i_1,\dots,i_s}(t)x_{i_1}\dots x_{i_s}. \tag{5}$$

For given available information sets $\eta_i(t)$ and a given set of strategies $\gamma^i \in \Gamma^i$ ($i \in \mathbb{N}$), the control actions are completely determined by the relations $u^i = \gamma^i(\eta_i)$. Substituting the set u^i into the cost functional (2) for a fixed final time T leads to the number $L_T^i(u^i, u^{-i})$, $i \in \mathbb{N}$, that is the cost incurred by player i defined in the control action space (Basar & Olsder, 1999). For fixed initial state x_0, we get the mapping defined by

$$J_T^i : \Gamma^1 \times \Gamma^1 \times \cdots \times \Gamma^N \mapsto \mathbb{R},$$
$$(\gamma^1, \gamma^2, \dots, \gamma^N) \mapsto L_T^i(u^i, u^{-i}),$$