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Brief paper

# A clipped-optimal control algorithm for semi-active vehicle suspensions: Theory and experimental evaluation[☆]

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ABSTRACT

This paper addresses the problem of optimal control for semi-active vehicle suspensions. A specific goal is to develop an algorithm which is capable of optimising ride and handling behaviour simultaneously in an experimental situation. A time-domain optimal control approach is adopted in which ride and handling are modelled as exogenous disturbances acting on the vehicle: road disturbances (modelled stochastically), and driver inputs (treated as deterministic quasi-static disturbances). A control algorithm is derived from a solution of the stochastic Hamilton–Jacobi–Bellman equation for the finite horizon case. The advantages of the approach are demonstrated experimentally on a test vehicle performing a steering manoeuvre on a bumpy roundabout.



## 1. Introduction

This paper is concerned with the design and experimental implementation of a clipped-optimal Linear Quadratic (LQ) semi-active suspension system. We focus on a suspension design framework which aims to insulate the body simultaneously from both road irregularities and handling disturbances (driver inputs, e.g. due to cornering, braking, etc.). Recent work on LQ semi-active suspension design – see for example Butsuen and Hedrick (1989), Du, Sze, and Lam (2005), Gordon (1995), Hrovat (1997), Savaresi, Poussot-Vassal, Spelta, Sename, and Dugard (2010), Sharp and Peng (2011) and Tseng and Hedrick (1994) and references therein – has often concentrated on the vehicle's response to road disturbances only. The incorporation of load disturbances into an LQ optimal control and estimation framework was proposed in Brezas and Smith (2013) in the context of active vehicle suspensions to deal with handling inputs. There, it was demonstrated, in simulation examples, that the use of a quasi-static model of the load forces is necessary both for effective control and to ensure good performance of the estimator. In the present work this approach is extended to the case of semi-active suspensions. We approach the optimal control problem by solving a stochastic Hamilton–Jacobi–Bellman (HJB) equation on a finite horizon, which motivates a constant gain clipped-optimal control law. This paper presents an experimental implementation of the algorithm on a prototype vehicle (made available as a test platform for this research) which clearly demonstrates the advantages of the approach (i.e., incorporating a model of the load disturbances in the control and estimator design). The vehicle was subjected to a slalom-type manoeuvre involving large steering inputs and significant road undulations. We provide a comparison with the standard LQG control in the literature (i.e., ignoring the load disturbance modelling), as well as a comparison with two fixed damping policies.

## 2. Quarter-car model and problem formulation

A typical semi-active suspension has a fixed spring $k_s$ in parallel with a rapidly adjustable damper with damping coefficient $u(t)$ that satisfies an inequality of the form

$$0 < c_{\min} \le u(t) \le c_{\max}. \quad (1)$$

As usual for the control design, we take the suspension spring to be linear and we approximate the tyre by a linear spring. The equations of motion are given by

$$m_s\ddot{z}_s = F_s - k_s(z_s - z_u) - u(\dot{z}_s - \dot{z}_u) \quad (2a)$$

$$m_u\ddot{z}_u = k_s(z_s - z_u) + u(\dot{z}_s - \dot{z}_u) - k_t(z_u - z_r), \quad (2b)$$

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where $z_s, z_u, z_r$ are the displacements of the sprung mass, unsprung mass and road, $m_s$, $m_u$, $k_s$, $k_t$ are the respective mass and spring constants. In this section we assume that $z_r$ is a Wiener process, i.e. $\dot{z}_r$ is Gaussian white noise. (For the full-car vehicle model we take a more realistic coloured noise road disturbance excitation.) As in Brezas and Smith (2013), Smith (1995) and Smith and Wang (2002) we include a load disturbance $F_s$ on the sprung mass to approximately model the effect of handling inputs. More precisely, $F_s$ effectively models the inertial forces induced by handling manoeuvres (such as cornering, braking, etc.) on the body, changes to static loads, as well as the aerodynamical loads and is treated deterministically. We can write the bilinear model in state-space form as

$$\dot{x} = Ax + BN^T xu + Fd + Gw, \tag{3}$$

where $x = [z_s - z_r, \dot{z}_s, z_u - z_r, \dot{z}_u]^T \in \mathbb{R}^4$, $u \in U \triangleq [c_{\min}, c_{\max}]$, $d = F_s \in \mathbb{R}$, and $w = \dot{z}_r \in \mathbb{R}$. Displacements are chosen relative to the road (rather than as absolute displacements) and the corresponding state-space matrices can be found in Brezas and Smith (2013). This has certain advantages in case the model is subjected to ramp inputs in $z_r$.

We make the common assumption that the adjustable damper can deliver the requested (admissible) damping instantaneously. Practical limitations would apply but these depend on the type of adjustable damper used (see Poussot-Vassal, Spelta, Sename, Savaresi, & Dugard, 2012 for more details). The reader is referred to Elmadany, Abduljabbar, and Foda (2003), Fialho and Balas (2002), Hac (1994), Hrovat (1990), Ray (1992), Ulsoy, Hrovat, and Tseng (1994), Williams and Haddad (1997), Wilson, Sharp, and Hassan (1986) and Youn, Im, and Tomizuka (2006) for further background on LQ active suspensions.

We consider the performance index

$$J = \mathbb{E}\left[\frac{1}{2}\int_0^T \left(q_0\ddot{z}_s^2 + q_1(z_s - z_r)^2 + q_2\dot{z}_s^2 + q_3(z_u - z_r)^2 + q_4\dot{z}_u^2 + ru^2\right) dt\right], \tag{4}$$

which is to be minimised over $u$. We take an initial condition $x_0$ which is a Gaussian random vector independent of $w$. We note that $J$ includes the sprung mass acceleration, tyre deflection and sprung mass displacement weightings (which are directly related to the main objectives), but also the sprung and unsprung velocity and control weightings which can in general be used for more flexibility in tuning (e.g. to achieve well-damped responses). We can write the performance index as

$$J = \mathbb{E}\left[\int_0^T l(x, u)dt\right], \tag{5}$$

where

$$l(x, u) = \frac{1}{2}\begin{bmatrix} x \\ u \\ d \end{bmatrix}^T \begin{bmatrix} Q & S_1 x & S_2 \\ x^T S_1^T & x^T R_1 x & S_3 x \\ S_2^T & x^T S_3^T & R_2 \end{bmatrix} \begin{bmatrix} x \\ u \\ d \end{bmatrix},$$

$S_1 = M_1 N^T$, $\quad S_2 = M_2$,

$S_3 = M_3 N^T$, $\quad$ and $\quad R_1 = RNN^T$.

The entries of $Q$, $M_1$, $M_2$, $M_3$, $R$ and $R_2$ can be found in Brezas and Smith (2013).

## 3. Clipped-optimal stationary control

In this section we provide a treatment of the optimal semi-active suspension control problem for the quarter-car model that also includes a deterministic load disturbance acting on the sprung mass. We first show that an optimal control exists, and subsequently we apply the sufficient conditions for optimality to obtain an optimal control.

### 3.1. Optimal control formulation

For $l(x, u)$ defined in Section 2 we formalise our optimal control problem as follows:

$$\boxed{\begin{array}{c} \text{Minimise } \mathbb{E}\left[\int_0^T l(x, u)dt\right] \\ \text{over measurable } u : [0, T] \to \mathbb{R} \\ \text{and loc. abs. continuous } x : [0, T] \to \mathbb{R}^4, \text{ s.t.} \\ \left\{ \begin{array}{c} \dot{x} = Ax + BN^T xu + Fd + Gw, \\ x(0) = \mathbb{E}[x_0], \ u(t) \in U. \end{array} \right\}. \end{array}} \tag{P}$$

In this section we assume that the full state $x$ is available for feedback. In Section 4.3 we describe the use of a Kalman filter to estimate the state for a full-car vehicle model.

### 3.2. Existence of an optimal control

**Lemma 1.** *The problem (P) has an optimal solution.*

**Proof.** It is straightforward to see that the conditions for existence of solutions in Fleming and Rishel (1975, Theorem 6.3, p. 170) are satisfied by the problem (P). □

### 3.3. Sufficient conditions for optimality

**Theorem 2.** *Consider the problem (P). Assume that, for a given initial state $x_0$, it is possible to find a control*

$$\bar{u} = sat_{[c_{\min}, c_{\max}]}\left\{-(N^T x)^{-1} R^{-1}\left[(B^T P + M_1)x - B^T \sigma + M_3 d\right]\right\}, \tag{6}$$

*and a solution to the following boundary value problem:*

$$\dot{x} = \begin{cases} (A + BN^T c_{\min})x + Fd, & \bar{u} = c_{\min} \\ (A - BR^{-1}(B^T P + M_1))x \\ \quad - BR^{-1}B^T\sigma + (F - BR^{-1}M_3)d, & \bar{u} \in (c_{\min}, c_{\max}) \\ (A + BN^T c_{\max})x + Fd, & \bar{u} = c_{\max} \end{cases}$$

*where $P(t)$ is a symmetric positive-definite matrix and $\sigma(t)$ a vector satisfying*

$$\dot{P} = \begin{cases} \phi_1(P), & \bar{u} = c_{\min} \\ \phi_2(P), & \bar{u} \in (c_{\min}, c_{\max}) \\ \phi_3(P), & \bar{u} = c_{\max} \end{cases} \tag{7}$$

$$\dot{\sigma} = \begin{cases} \psi_1(\sigma), & \bar{u} = c_{\min} \\ \psi_2(\sigma), & \bar{u} \in (c_{\min}, c_{\max}) \\ \psi_3(\sigma), & \bar{u} = c_{\max} \end{cases} \tag{8}$$

*where*

$$\phi_1(P) = -P(A + BN^T c_{\min}) - (A + BN^T c_{\min})^T P - Q - 2M_1 N^T c_{\min} - Rc_{\min}^2,$$

$$\phi_2(P) = -P(A - BR^{-1}M_1^T) - (A - BR^{-1}M_1^T)^T P + PBR^{-1}B^T P - Q + M_1 R^{-1} M_1^T,$$

$$\phi_3(P) = -P(A + BN^T c_{\max}) - (A + BN^T c_{\max})^T P - Q - 2M_1 N^T c_{\max} - Rc_{\max}^2,$$

$$\psi_1(\sigma) = -(A^T + NB^T c_{\min})\sigma - (PF + M_3 N^T c_{\min} + M_2)d,$$

$$\psi_2(\sigma) = -\left[A^T - M_1 R^{-1} B^T - PBR^{-1}B^T\right]\sigma - \left[M_2 - M_1 R^{-1} M_3 + P(F - BR^{-1}M_3)\right]d,$$

$$\psi_3(\sigma) = -(A^T + NB^T c_{\max})\sigma - (PF + M_3 N^T c_{\max} + M_2)d,$$

*with boundary conditions $x(0) = x_0$, $P(T) = 0$, and $\sigma(T) = 0$. Then, $\bar{u}$ is an optimal control.*