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Brief paper

Practical synchronization with diffusive couplings☆

Jan Maximilian Montenbruck, Mathias Bürger, Frank Allgöwer

Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany

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ABSTRACT

We investigate the problem of synchronizing nonidentical or perturbed nonlinear systems. In the considered setup, the systems are incapable to synchronize under diffusive couplings. Instead, assuming the QUAD property for each system, we derive conditions under which the synchronization error can be kept arbitrarily small by a proper choice of the interconnection structure. This motivates the definition of practical synchronization as an alternative synchronization notion for nonidentical or perturbed dynamical systems. The presented results are intimately related to synchronization of passive systems, but it is shown that the stronger QUAD assumption is essential in our framework. The proposed concept of practical synchronization translates directly into a notion of robust synchronization. Beyond that, the results open the way for an investigation of synchronization phenomena on unbalanced graphs, leading to the concept of cluster synchronization.

1. Introduction

Pecora and Carroll (1990) observed synchronization experimentally in diffusively coupled circuits, showing that approximate synchronization with arbitrarily small ultimate bounds on the synchronization error can be enforced solely with diffusive couplings, even if parameter mismatches between the systems are introduced. These observations suggest that, for certain systems, diffusive couplings are sufficient for approximate synchronization with arbitrary precision, even if the systems are nonidentical ("heterogeneous"). In this paper, we investigate conditions under which diffusive couplings are sufficient to achieve approximate synchronization and that in this case, the synchronization error depends on the heterogeneity of the systems and on the coupling strength. In particular, the less the systems differ from each other, the lower the ultimate bound on the synchronization error shall become and any arbitrarily small ultimate bound on the synchronization error can be achieved by choosing the diffusive couplings appropriately, motivating the notion of practical synchronization.

Related work. Several approaches aim to achieve synchronization of nonidentical systems by imposing local controllers to the individual systems: adaptive local controllers were applied for synchronization of systems with uncertainties (Li & Chen, 2004; Zhou, Lu, & Lü, 2006) and it is possible to achieve impulsive synchronization of uncertain systems with local control schemes (Liu, Liu, Chen, & Wang, 2005). Moreover, systems are practically synchronizable by local controllers (Femat & Solís-Perales, 1999; Sekieta & Kapitaniak, 1996) or external signals (Brucoli, Cafagna, & Carnimeo, 2001). The synchronous solution of double-integrators (Carli & Lovisari, 2012) and linear systems (Trentelman, Takaba, & Monshizadeh, 2013) is inherently robust (with respect to heterogeneities and uncertainties, respectively). Other approaches investigate the synchronization of identical systems with diffusive couplings, but assume certain systems properties, such as passivity: relaxed cocoercivity (Scardovi, Arcak, & Sontag, 2010), incremental passivity (Hamadeh, Stan, & Sepulchre, 2012), passivity (Arcak, 2007; Ihle, Arcak, & Fossen, 2007), and semipassivity (quasipassivity) (Pogromsky & Nijmeijer, 2001) are sufficient for synchronization, all among identical systems. Error-passivity is also sufficient for synchronization in nonidentical systems (Yao, Guan, & Hill, 2009). The QUAD condition is a sufficient condition for synchronization of identical systems (DeLellis, di Bernardo, & Garofalo, 2008, 2009; DeLellis, di Bernardo, & Russo, 2011; Liu, Cao, & Wah Wu, 2014; Liu & Chen, 2008) whereas necessary conditions for synchronization were derived in terms of internal models (Wieland, Wu, & Allgöwer, 2013; Zhao, Hill, & Liu, 2011). In this

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E-mail addresses: jan-maximilian.montenbruck@ist.uni-stuttgart.de (J.M. Montenbruck), mathias.buerger@ist.uni-stuttgart.de (M. Bürger), frank.allgower@ist.uni-stuttgart.de (F. Allgöwer).

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last contribution, as well as in recent work (Grip, Yang, Saberi, & Stoorvogel, 2012; Kim, Shim, & Seo, 2011), synchronization of nonidentical systems that are diffusively coupled via their outputs is analyzed. Synchronization of nonidentical systems is possible by solely applying diffusive couplings, but using infinite gains (Hale, 1997). Recently, the aforementioned results on synchronization of systems satisfying the QUAD property were extended to approximately synchronizing nonidentical systems with possibly non smooth vector fields and nonlinear output couplings under the assumption that the vector fields can be decomposed into a common part and a bounded heterogeneous perturbation (DeLellis, di Bernardo, & Liuzza, 2014).

Contributions. We study synchronization phenomena in systems which do not satisfy the necessary conditions for synchronization. We provide sufficient conditions for diffusively coupled systems to synchronize approximately for arbitrarily small positive ultimate bounds on the synchronization error. In particular, we show that for every chosen ultimate bound $\epsilon > 0$ on the synchronization error, there exists a (finite) diffusive coupling such that the synchronization error is ultimately bounded by ϵ. While (incremental) passivity is sufficient for synchronization among identical systems and provides insight into the synchronization problem for nonidentical systems, we will need to impose assumptions, in particular the QUAD property, on the vector fields of the systems. The presented result is both, constructive and analytic, as it yields sufficient conditions on a diffusive coupling to achieve a desired ultimate bound of the synchronization error as well as sufficient conditions for approximate synchronization. The main result on practical synchronization can be applied to two concepts that appear to be relevant on their own; first, we show that our result applies to robust synchronization, where practical synchronization can be shown for a whole class of uncertainties and second, our result allows to investigate cluster synchronization on general directed graphs, whereas undirected or weight-balanced graphs are assumed in most research on synchronization.

Organization. The remainder of the paper is organized as follows; in Section 2, we specify the problem statement. In Section 3, we review a particular result on the nominal case of our setup. We then present our main result in Section 4. The aforementioned two extensions of the latter are outlined in Section 5. Our findings are illustrated on a numerical example in Section 6 and Section 7 concludes the paper.

Notation. For $z \in \mathbb{R}^{n\times m}$, we write $z = \left[z_{ij}\right]$ to clarify that z_{ij} denotes the ith element of the jth column of z. A function that maps $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is said to be of class $\mathcal{KL}$, if it is zero at zero and strictly increasing in the first argument and decreasing to zero in the second argument. I_n denotes the identity of $\mathbb{R}^{n\times n}$ and $1_n = [1 \cdots 1]^\top$ is the n-fold vector of ones (or Fiedler vector). The direct product of matrices (or Kronecker product) is written as $\otimes$. Similarly, $\oplus$ denotes not only the direct sum of matrices, but also the direct sum of vector spaces (the distinction can be inferred from the context). $\emptyset$ is an empty set and $\ker M$ denotes the nullspace of the matrix M.

2. Problem statement

In this paper, we consider a collection of N systems of the form

$$\dot{x}_i = f(x_i) + \Delta_i(x_i) + u_i + w_i \tag{1}$$

where $i \in \{1 \cdots N\}$, $\Delta_i : \mathbb{R}^n \to \mathbb{R}^n$ models the heterogeneity of the systems, and $w_i : \bar{\mathbb{R}}^+ \to \mathbb{R}^n$ is an external signal, representing, e.g., a perturbation. The vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be known and identical for all i. Both Δ_i and f are assumed to be at least continuous and such that the solution $\varphi_i : \mathbb{R}^n \times (-\epsilon, \epsilon) \to \mathbb{R}^n$, $\left(x_{i,0}, t\right) \mapsto \varphi_i\left(x_{i,0}, t\right)$ to (1) exists and is unique at least on some interval $(-\epsilon, \epsilon)$. The coupling of the systems will be realized through the control input u_i. We suppose that the N systems (1) are linearly coupled via

$$u_i = -\sum_{j=1}^{N} L_{ij} x_j, \tag{2}$$

where $L = \left[L_{ij}\right] \in \mathbb{R}^{N\times N}$ is the coupling matrix, to be designed later on. For the overall system, we frequently use the "stacking" notation

$$\begin{aligned}\dot{x} &= F(x) + \Delta(x) + u + w \\ &= F(x) + \Delta(x) - (L \otimes I_n)\, x + w,\end{aligned} \tag{3}$$

where $\Delta : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is the stack of Δ_i, $F : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ the N-fold stack of f, u the stack of u_i, and w the stack of w_i. We denote the solution to (3) by $\varphi : \mathbb{R}^{Nn} \times (-\epsilon, \epsilon) \to \mathbb{R}^{Nn}$, $(x_0, t) \mapsto \varphi(x_0, t)$ (i.e. φ is the stack of φ_i and x_0 is the stack of $x_{i,0}$). In what follows, we assume that L is a so-called diffusive coupling, just as it was motivated in Section 1. This is the case when its diagonal elements are strictly positive, its off-diagonal elements are nonpositive, and it has a eigenvector of ones associated with a null eigenvalue.

Assumption 1. For the coupling matrix L, $L_{ij} \le 0$ for $i \ne j$ and $L1_N = 0$ hold true.

With L chosen such that it satisfies Assumption 1, we can interpret it as a Laplacian matrix and assign a unique weighted, directed graph G with vertices $\{1 \cdots N\}$ to it such that $E = \{(i,j)\,|\, i \ne j, L_{ij} < 0\}$ determines its edges and $-L_{ij}$ is the weight on the edge (i,j). In the remainder of this paper, when we refer to the graph associated with L, we mean G as defined above. We are interested in phenomena where the systems converge to the average of their solutions $s : \mathbb{R}^{Nn} \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ given by

$$s(x_0, t) = \frac{1}{N}\sum_{i=1}^{N} \varphi_i\left(x_{i,0}, t\right),$$

or at least stay close to it; i.e. to make the synchronization errors $e_i : \mathbb{R}^{Nn} \times (-\epsilon, \epsilon) \to \mathbb{R}^n$

$$e_i(x_0, t) = \varphi_i\left(x_{i,0}, t\right) - s(x_0, t)$$

go to zero asymptotically, or to at least keep them small by means of appropriately choosing L. The former is called synchronization. If in the latter case, the ultimate bound for the error can be kept arbitrarily small by a proper choice of L, this is called practical synchronization. In the case $\Delta_i = 0$ and $w = 0$, synchronization is possible. Without these assumptions, merely using diffusive couplings (and no local controllers), only practical synchronization is possible. In the following, in the spirit of the above stacking notation, e will denote the stack of e_i.

Definition 1. The systems (1) under coupling (2) are said to synchronize, if for all $x_0 \in \mathbb{R}^{Nn}$ $\|e(x_0, t)\| \le \beta(\|e(x_0, 0)\|, t)$, where β is a class $\mathcal{KL}$ function.

Definition 2. The systems (1) under coupling (2) are said to be practically synchronizable, if, for every positive choice of ϵ, there exists L satisfying Assumption 1 such that for all $x_0 \in \mathbb{R}^{Nn}$, $\|e(x_0, t)\| \le \beta(\|e(x_0, 0)\|, t) + \epsilon$ is satisfied, where β is a class $\mathcal{KL}$ function.

As the graph associated to L is unique and vice versa, every graph defines a unique L, the definition could equivalently read: "for every positive choice of ϵ, there exists a directed, weighted graph". For couplings satisfying Assumption 1, we can easily derive a necessary condition for synchronization. The necessary condition is well-known and has been investigated in particular in Zhao et al. (2011), Wieland et al. (2013) and Bürger and De Persis (2013).