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Brief paper

Non-linear pricing by convex duality☆

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ABSTRACT

We consider the pricing problem of a risk-neutral monopolist who produces (at a cost) and offers an infinitely divisible good to a single potential buyer that can be of a finite number of (single dimensional) types. The buyer has a non-linear utility function that is differentiable, strictly concave and strictly increasing. Using a simple reformulation and shortest path problem duality as in Vohra (2011) we transform the initial non-convex pricing problem of the monopolist into an equivalent optimization problem yielding a closed-form pricing formula under a regularity assumption on the probability distribution of buyer types. We examine the solution of the problem when the regularity condition is relaxed in different ways, or when the production function is non-linear and convex. For arbitrary type distributions, we offer a complete solution procedure.

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1. The setting

Non-linear pricing is a basic problem of economic mechanism design under asymmetric information. Consider a monopolist who is producing an infinitely divisible good, e.g., sugar, and wishes to sell the good to a potential buyer with unknown valuation for his/her product. The seller's production function is assumed to be linear with a slope equal to $c > 0$. The seller is risk neutral, and therefore, seeks to maximize the expected revenue from the sale. The buyer can be one of a finite number of types t from the index set $\mathcal{T} = \{1, \ldots, m\}$ with $m > 2$. The parameter t for the type of the buyer is assumed to represent the valuation of a potential buyer for the good. The buyer derives a utility equal to $t \cdot u(\mathcal{A}_t) - p_t$ from acquisition of a quantity $\mathcal{A}_t$ (allocation to buyer of type t) of the good, where u is a differentiable, strictly concave, strictly increasing function ($u''(x) < 0, u'(x) > 0$ for every x) with $u(0) = 0$ and a strictly decreasing $(u')^{-1}$, and p_t is the price paid for acquisition of the quantity $\mathcal{A}_t \geq 0$. The crux of the problem is that a potential buyer's type (or valuation of the good) t is private, i.e., unknown to the seller. However, the seller's beliefs about t are given by a probability mass function f on the discrete set $\mathcal{T}$. The problem of the seller is to devise a mechanism that will maximize expected revenue while it elicits a truthful declaration of type by the seller and ensures his/her participation.

The non-linear pricing problem briefly described above occurs in many industries, e.g., wireless communication services, other telecom and technology products, legal plans, fitness clubs, automobile clubs and healthcare plans; see Bagh and Bhargava (2013) for further details. It is part of the general theory of basic static adverse selection problems in economics. The study of the problem was started in Mirrlees (1971) and developed into a mature subject with numerous contributions (a notable one is the paper by Myerson, 1981) that would be impractical to cite in this short note. An authoritative and detailed reference on nonlinear pricing is Wilson (1997).[1] As it is closer to our treatment, we adopt as our desktop reference on non-linear pricing of a single good the book by Bolton and Dewatripont (2004) which contains a list of the main references on the subject up to 2005. One can find in Chapter 2 of Bolton and Dewatripont (2004) discussions of the non-linear pricing problem first with two types, and then with a finite number of different types and then, a continuum of types using methods that are different from that of the present note. In fact, the Ref. Bolton and Dewatripont (2004) does not offer an explicit solution for the case of discrete types while (nor does Wilson, 1997 for that matter) for a continuum of types a closed-form pricing formula (credited to Baron & Myerson, 1982 and Maskin & Riley, 1984) is given under a condition on the utility function and the monotonicity assumption on the probability distribution of types. When the monotonicity assumption is violated, a so-called *ironing* procedure gives the optimal contract with a *bunching/pooling* property (the optimal

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[1] Our utility model differs from that of Wilson where the dependence of buyer utility on type is not made explicit.

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allocation remains constant over some interval) using the methods of calculus of variations. Other noteworthy references include Champsaur and Rochet (1989), Figueroa and Skreta (2007), Guesnerie and Seade (1982), Matthews and Moore (1987) and Moore (1984). In Figueroa and Skreta (2007), the auction of multiple goods is considered for the case of a continuum of types where preferences are represented by a non-linear utility function. It is shown that when incentive compatibility constraints bind, a *randomized* mechanism may be optimal as opposed to the deterministic mechanisms considered in the present paper. An interesting application of non-linear pricing where monotonicity assumption may be violated is reported in Crawford and Shum (2007) where the authors explore the degree of quality degradation in cable television markets and the impact of regulation on those choices using empirical data from cable networks. Taking the utility function of consumers to be linear in quality, they utilize two, three or four types of consumers in the model of monopoly choice of Mussa and Rosen (1978) which addresses the problem of a monopolist selling two goods whose qualities varies over a finite interval to consumers that are differentiated by a parameter that can take distinct values where the first type represents consumers who prefer not to purchase any of the cable network products. The empirical consumer type distributions derived from market share data may indeed violate monotonicity (cf. Table 5, p. 201 of Crawford & Shum, 2007).

Against this background, the purpose of the present note is to derive a simple explicit price formula for the case of discrete types using the machinery of convex optimization and duality as advocated by Vohra (2011, 2012) although the mechanism design problem is initially formulated as a non-convex optimization problem. The contribution of the manuscript is to bring to bear the novel analysis technique based on convex duality of Vohra on instances where regularity of the types distribution is violated. The main results are the discrete-types analogs of the continuous types results of the literature. Our first result is obtained under a regularity (monotonicity) assumption of the probability mass f as in Bolton and Dewatripont (2004). The result extends in a straightforward manner to the case of convex production cost function of the monopolist. Then, we relax gradually the regularity assumption and prove further results for the optimal mechanism which mimics the *ironing/bunching(pooling)* solution of the continuous types case. To the best of our knowledge, the present note is one of the few papers that addresses the discrete (single dimensional) multiple types (more than two types) non-linear pricing problem from a mathematical programming perspective along with e.g., Bandi and Bertsimas (2012) and Vohra (2012). This short note may also serve as an entry point for newcomers to the subject as it treats a simpler setting and uses rather basic tools of optimization, compared to e.g., Vohra (2012) which involves optimization over poly-matroids. An important feature of our paper is that any instance of the non-linear pricing problem described in the present paper can be solved explicitly without resorting to a non-linear optimization software. We illustrate our results with examples.

By virtue of the Revelation Principle (Vohra, 2011), the seller is interested in designing a direct mechanism that consists of the two discrete functions p (for price) and $\mathcal{A}$ (for allocation), both functions of type t. In other words, the seller implementing a direct mechanism declares a price p_t and a quantity allocation $\mathcal{A}_t$ for each type t. Against this background, the problem of pricing the indivisible good is formulated as the following optimization problem. We define the decision variables p_t for all $t \in \mathcal{T}$ for the price quoted by the seller to a buyer of type t, in addition to the non-negative allocation variables $\mathcal{A}_t$. The seller wishes to maximize the expected profits from the sale:

$$\sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t) \tag{1}$$

under the restrictions of Incentive Compatibility (IC) and Individual Rationality (IR) that are, respectively:

$$t(u(\mathcal{A}_t) - u(\mathcal{A}_s)) \geq p_t - p_s, \quad \forall t, s \in \mathcal{T} \tag{2}$$

$$t \cdot u(\mathcal{A}_t) - p_t \geq 0, \quad \forall t \in \mathcal{T}. \tag{3}$$

The constraint (IC) ensures that the utility of the seller that declares his/her type truthfully is at least as large as the utility derived from reporting a different type. The constraint (IR) is to ensure that the minimum (reservation) utility of any buyer of any type is at least zero, which leads to ensuring participation of the buyers into the mechanism.

Therefore, the seller seeks a pair $p_t, \mathcal{A}_t \geq 0$ for each type $t \in \mathcal{T}$ that maximizes (1) under the restrictions (2)–(3). Note that the problem (1)–(2)–(3) is in general non-convex due to the presence of the difference $u(\mathcal{A}_t) - u(\mathcal{A}_s)$ which is not necessarily a concave function. In the next section we prove a simple result departing from hidden convex (more precisely, concave since we are maximizing) structure in the problem.

2. The optimal mechanism under monotonicity

Let $\nu_t = t - \frac{1-F_t}{f_t}$ for all $t \in \mathcal{T}$ where we denote by F the cumulative distribution function associated with the mass function f (ν_t is commonly referred to as the *virtual valuation*). The economic meaning attached to the virtual valuation of the bidder is the marginal revenue obtained by allocating the item to this bidder. As is common to most references, see e.g., Bolton and Dewatripont (2004), Tirole (1990) and the references therein, we assume ν_t to be monotone increasing in t. We call f *regular* if the m-vector ν associated with f is monotone increasing.[2] A way to enforce the above monotonicity is the so-called Monotone Hazard Rate (MHR) condition. A distribution F with density f is said to satisfy the MHR condition if the *hazard rate* $\frac{f(t)}{1-F(t)}$ is non-increasing with t. Most well-known continuous distributions satisfy the MHR condition, e.g., the uniform, the normal, the Pareto, the logistic, the exponential; see Section 3.5 of Tirole (1990). Therefore, one may safely assume that it will hold for their discretized counterparts.

The first result of the note is the following.

Proposition 1. *For regular f there exists an optimal direct mechanism with the allocation*

$$\mathcal{A}_t^* = (u')^{-1}\left(\frac{c}{\nu_t}\right), \tag{4}$$

and the optimal prices

$$p_t^* = t \cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*) \tag{5}$$

for all $t = t^*, \ldots, T$ where t^* is the smallest value of t that satisfies: $\nu(t^*) > 0$, and $\mathcal{A}_t^* = p_t^* = 0$ for all $t = 1, \ldots, t^* - 1$.

Proof. We can always define a dummy type $t = 0$ with $\mathcal{A}_0 = p_0 = 0$ and incorporate constraint (3) into (2); see Vohra (2011). Then, we re-write the problem (1)–(2)–(3) as the following convex optimization problem

$$\max_{p_t, y_t, \mathcal{A}_t \geq 0} \sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t)$$

subject to

$$t(y_t - y_s) \geq p_t - p_s, \quad \forall t, s \in \mathcal{T}$$

$$u(\mathcal{A}_t) \geq y_t, \quad \forall t \in \mathcal{T}.$$

[2] In fact, it is sufficient that the positive components of ν are monotone increasing.

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