Automatica 51 (2015) 18–26

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A dual pair of optimization-based formulations for estimation and control[☆]

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ARTICLE INFO

Article history:
Received 18 August 2013
Received in revised form
7 April 2014
Accepted 6 October 2014

Keywords:
Duality
Deadbeat control
Deadbeat observer
Moving horizon estimation
Receding horizon control

ABSTRACT

A finite-horizon optimal estimation problem for discrete-time linear systems is formulated and solved. The formulation is a natural extension of that which yields a deadbeat observer. The resultant observer is the dual of the controller produced by the finite-horizon minimum energy control problem with terminal equality constraint. Nonlinear extensions of this dual pair are also considered and sufficient conditions are provided for stability and convergence.

1. Introduction

One of the earliest things that students of control theory are taught is that for linear systems controllability and observability are *dual* concepts. Very few doubt this because it is in every linear systems textbook. Interestingly, what is usually not in all those books is a clear definition of duality (Luenberger, 1992). A possibility is that no one wants to confine the notion into the precision required by a definition. Or, perhaps, it is too obvious a thing to define. Either way, people do not seem to need its exact description in order to make use of or enjoy duality; for once a dual pair emerges, the human eye is very quick to recognize it.

An intriguing example of duality is between the problems of linear quadratic regulation (LQR) and linear quadratic estimation (LQE, Kalman–Bucy filter). These celebrated optimization problems, which are very different conceptually and formulation-wise, yield sets of parameters (matrices) that are associated via formal rules that transform one set to another Kalman (1960).[2] The problems of linear deadbeat control and linear deadbeat estimation make another example of a dual pair. Let us recall the former. Consider the below systems, both nth order,

$$x_{k+1} = Ax_k \tag{1}$$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k \tag{2}$$

where the system (2) is to track the system (1) by choosing suitable control inputs $u_0, u_1, \ldots$. (Let us assume for now that the controllability condition is satisfied, input u is scalar, and the full state information $(\hat{x}, x)$ of both systems is available to the controller.) To turn the system (2) into a deadbeat tracker for the system (1), i.e., to achieve $\hat{x}_k = x_k$ (for arbitrary initial conditions $\hat{x}_0, x_0$) for $k \geq n$, one can follow the below method, which, although stated for linear systems, is equally meaningful for nonlinear systems.

Algorithm 1. Apply u_k from the sequence of inputs $(u_k, u_{k+1}, \ldots, u_{k+n-1})$ obtained by solving $\hat{x}_{k+n} = A^n x_k$.

If we now move to the dual problem, linear deadbeat estimation, how to translate Algorithm 1 is not immediately clear. Motivated by the historical pattern that beautiful things tend to come in dual pairs for linear systems, our work here starts with a search for this missing twin of Algorithm 1. In more exact terms, guided by linear duality, we look for some sort of a principle that not only leads to linear deadbeat observer but also is useful for the nonlinear deadbeat observer design. This search is nothing but a simple linear algebra exercise, but its outcome turns out to have some interesting consequences that go beyond *linear* and *deadbeat*. Those

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dragan Nesic under the direction of Editor Andrew R. Teel.

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[2] Though LQR and LQE are acknowledged as a dual pair, nowhere (to the best of our knowledge) it is mentioned whether duality played much (if any) role in their discoveries. In other words, there seems to be no evidence to suggest that the birth of LQE was a consequence of the pressing fact that LQR must have a *twin*.

http://dx.doi.org/10.1016/j.automatica.2014.10.110

consequences are what we report in this paper. In particular, three things are done.

First, in Theorem 4 observer design for linear systems is formulated as a finite-horizon optimization problem. The formulation concerns a moving-horizon type observer (whose order matches that of the system being observed) where at each time an estimate of the system state is generated based solely on the current output (instead of a larger collection of data comprising previous measurements) of the system and the current observer state. Convergence is guaranteed for all horizon lengths no smaller than the observability index of the system being observed. Interestingly, the formulation presented here turns out to be the dual of a classic result (Theorem 5) by Kleinman (1974), who is acknowledged to be the first to consider moving-horizon feedback (Keerthi & Gilbert, 1988). Note that our formulation is in discrete-time. The continuous-time formulation is due to Thomas (1975).

Second, in Theorem 9 a nonlinear generalization of the linear optimal observer construction of Theorem 4 is provided, where convergence is established under certain conditions inspired by those that hold in the linear problem. The resulting nonlinear moving-horizon observer, like its above-mentioned linear version, is driven only by the current output value of the system being observed. This constitutes an apparent conceptual difference between the construction in this paper and the existing work on moving-horizon estimation, the basic philosophy of which is summarized in Ferrari-Trecate, Mignone, and Morari (2002) as: *the estimates of the states are obtained by solving a least squares problem, which penalizes the deviation between measurements and predicted outputs of a system. The data considered for the optimization is laying in a window of fixed finite length, which slides forward in time.* Namely, at each time k the formulation presented here only requires the most recent (single) measurement y_k to run the optimization problem, whereas the literature, to the best of our knowledge, has so far only reported results, where the optimization is performed processing a collection of recent measurements $(y_{k-N+1}, \ldots, y_{k-1}, y_k)$, where N is no smaller than the observability index. See, for instance, Alessandri, Baglietto, and Battistelli (2008, 2012), Psiaki (2013) and Rao, Rawlings, and Mayne (2003).

Third, for the sake of symmetry we present in Theorem 12 a possible nonlinear extension of Kleinman's optimal controller (Theorem 5). More specifically, a moving-horizon optimal tracking problem is considered, where convergence is established mainly through terminal equality constraint. Though widely-used, terminal equality constraint is not indispensable and other means to achieve convergence have long existed in the receding horizon control literature. What has been indispensable for asymptotic stability however is some sort of detectability assumption[3] on the stage cost (Grimm, Messina, Tuna, & Teel, 2005; Limon, Alamo, Salas, & Camacho, 2006; Mayne, Rawlings, Rao, & Scokaert, 2000; Reble & Allgower, 2012). We note that in the very first (and linear) result on moving-horizon feedback (Kleinman, 1970), where the stage cost is a quadratic function solely of the control input, this assumption is violated. The novelty of the formulation in this paper is that we do not assume detectability from the stage cost nor that detectability is implied by our assumptions. We hence intend to provide a proper generalization of Kleinman (1974).

2. Notation

$\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. For a mapping $f : \mathcal{X} \to \mathcal{X}$ let $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. The Euclidean norm in $\mathbb{R}^n$ is denoted by $\|\cdot\|$. For a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ the smallest and largest eigenvalues of Q are respectively denoted by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$. Also, $\|x\|_Q^2 = x^T Qx$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, strictly increasing, and unbounded.

[3] It is true that recent works on economic model predictive control (Angeli, Amrit, & Rawlings, 2012; Grüne, 2013) shed this assumption, yet convergence in those works is still established through a *rotated* stage cost from which the state of the system is detectable thanks to some strict dissipativity assumption.

3. An optimal observer

We begin this section by an attempt to obtain the dual of Algorithm 1, i.e., some method to construct the deadbeat observer, which is meaningful also for nonlinear systems. Consider the discrete-time linear system

$$x^+ = Ax, \qquad y = Cx \tag{3}$$

where $x \in \mathbb{R}^n$ is the *state*, $y \in \mathbb{R}^m$ is the *output*, and x^+ is the state at the next time instant. The matrices A and C belong to $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{m\times n}$, respectively. We will denote by x_k (for $k \in \mathbb{N}$) the solution of the system (3) starting from the initial condition x_0. Driven by the output y of the system (3) and having started from the initial condition z_0, suppose that the below system, for $N \geq 1$,

$$z^+ = A\eta(z, y) \tag{4}$$

produces at each time k an estimate z_k of x_{k-N+1} (the $N-1$ steps earlier value of the current state x) based on z_{k-1} and y_{k-1}. That is, the vector $\eta \in \mathbb{R}^n$ is a function of the state z and the output y. Note that the system (4) can be used in the following observer

$$z^+ = A\eta, \qquad \hat{x} = A^{N-1}z \tag{5}$$

where $\hat{x}$ is the estimate of the current state x. Assuming for now that the system (3) is observable and its output y is scalar, we ask the following question. How should η be chosen such that the system (5) is a deadbeat observer for the system (3), i.e., $\hat{x}_k = x_k$ for $k \geq n$ regardless of the initial conditions z_0, x_0?

To answer the question we recall the deadbeat tracker, the dual of deadbeat observer. From Algorithm 1 it follows that the dynamics of the deadbeat tracker read $\hat{x}^+ = A\hat{x} + BK(x - \hat{x})$ with the feedback gain $K = e_n^T \mathcal{C}^{-1}A^n$ where $\mathcal{C} = [B \quad AB \quad \cdots \quad A^{n-1}B]$ is the controllability matrix and $e_n = [0 \quad \cdots \quad 0 \quad 1]^T$. By duality the dynamics of the deadbeat observer should read

$$\hat{x}^+ = A\hat{x} + L(y - C\hat{x}) \tag{6}$$

with the observer gain

$$L = A^n \mathcal{O}^{-1} e_n \tag{7}$$

where $\mathcal{O} = [C^T \quad A^TC^T \quad \cdots \quad A^{(n-1)T}C^T]^T$ is the observability matrix. Now, combining (5), (6), and (7) we can write

$$\begin{aligned} A^N\eta &= A^{N-1}z^+ = \hat{x}^+ \\ &= A\hat{x} + A^n\mathcal{O}^{-1}e_n(y - C\hat{x}) \\ &= A^Nz + A^n\mathcal{O}^{-1}e_n(y - CA^{N-1}z). \end{aligned}$$

If we let $N = n$ we can write

$$A^n\eta = A^n(z + \mathcal{O}^{-1}e_n(y - CA^{n-1}z))$$

which suggests that we choose η as

$$\eta = z + \mathcal{O}^{-1}e_n(y - CA^{n-1}z). \tag{8}$$

Eq. (8) is not directly generalizable to nonlinear systems so we rewrite it as the following set of equations

$$\left.\begin{aligned} &CA^i\eta = CA^iz \quad \text{for } i = 0, 1, \ldots, N-2 \\ &CA^{N-1}\eta = y \end{aligned}\right\} \tag{9}$$

keeping in mind that $N = n$. Therefore, to turn the system (5) (for $N = n$) into a deadbeat observer for the system (3) with scalar output one can use the below algorithm.