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Brief paper

Boundary feedback stabilization of the Schlögl system[☆]

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ABSTRACT

The Schlögl system is governed by a nonlinear reaction–diffusion partial differential equation with a cubic nonlinearity that determines three constant equilibrium states. It is a classical example of a chemical reaction system that is bistable. The constant equilibrium that is enclosed by the other two constant equilibrium points is unstable.

In this paper, Robin boundary feedback laws are presented that stabilize the system in a given stationary state or more generally in a given time-dependent desired system orbit. The exponential stability of the closed loop system with respect to the L^2-norm is proved. In particular, it is shown that with the boundary feedback law the unstable constant equilibrium point can be stabilized.

1. Introduction

The Schlögl system has been introduced in Schlögl (1972)as a model for chemical reactions for non-equilibrium phase transitions. It describes the concentration of a substance in 1-d. In neurology, the same nonlinear reaction–diffusion system is known under the name Nagumo equation and models an active pulse transmission through an axon (Chen & Guo, 1992; Nagumo, 1962). It is also known as Newell–Whitehead–Segel equation (see Newell & Whitehead, 1969 and Segel, 1969). This system is governed by a parabolic partial differential equation with a cubic nonlinearity that determines three constant equilibrium states $u_1 < u_2 < u_3$, where u_2 is unstable. In view of its simplicity, the Schlögl system may serve as a test case for the stabilization of an unstable equilibrium for reaction–diffusion equations that generate traveling waves. While this task might appear a little bit academic for the Schögl model, it is of paramount importance for more complicated equations such as the bidomain system in heart medicine, cf. Kunisch and Wagner (2013). Here, the goal of stabilization is to extinguish undesired spiral waves as fast as possible and hereafter to control the system to a desired state. However, there are similarities between these models and it is therefore reasonable to consider the same problem for the Schögl system.

The control functions can act in the domain (distributed control) or on its boundary. In this paper, the problem of boundary feedback stabilization is studied. Example 1 illustrates that, without the influence of the boundary conditions, the system state approaches exponentially fast a stable equilibrium, even if the initial state is arbitrarily close to the unstable equilibrium.

Also, the more general case of boundary stabilization of time-dependent states of the system is considered in this paper. This includes the stabilization of periodic states that is interesting as a tool to stabilize the periodic operation of reactors, see Silverston and Hudgins (2013). This case also includes the stabilization of traveling waves.

In this paper, linear Robin-feedback laws are presented that yield exponential stability with respect to the L^2-norm for desired orbits of the system. The term desired orbit is used to describe a possibly time-dependent solution of the partial differential equation that defines the system. The exponential stabilization is particularly interesting since the boundary feedback allows to stabilize the system in the unstable equilibrium that is enclosed by the other two constant equilibrium points.

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To show that the system is exponentially stable, we construct a strict Lyapunov function. The construction of strict Lyapunov functions for semilinear parabolic partial differential equations has also been studied in Mazenc and Prieur (2011). In Mazenc and Prieur (2011), it is assumed that the feedback is space-periodic or the boundary conditions are chosen in such a way that the product of the state and the normal derivative vanishes at the boundary. This assumption implies that the boundary terms that occur after partial integration in the time derivative of the Lyapunov function become nonpositive.

For the state feedback that is presented in this paper, this assumption does not hold. Therefore a different approach is used in the analysis: The Poincaré–Friedrichs inequality is used to show that the Lyapunov function is strict. Note that the Poincaré–Friedrichs inequality is often used to prove the existence or uniqueness of the solution of partial differential equations. However, to our knowledge, up to now it has not been used to show that a Lyapunov function is strict.

The Schlögl system has the interesting property that it allows traveling wave solutions (i.e. uniformly translating solutions moving with a constant velocity) that have the shape of the hyperbolic tangent (see Karzari, Lemarchand, & Mareschal, 1996). The traveling wave solutions connect the two stable constant stationary states. The problem to steer associated wave fronts to rest by *distributed* optimal control methods was considered in Buchholz, Engel, Kammann, and Tröltzsch (2013) for the Schlögl model and in Casas, Ryll, and Tröltzsch (2013) for the FitzHugh–Nagumo system, where spiral waves occur. In the present paper, we propose a *boundary* control law that stabilizes the system exponentially fast to a desired orbit.

The boundary control of a linear heat equation via measurement of domain-averaged temperature has been studied in Bošković, Krstić, and Liu (2001), Weijiu (2003). Results about the control of parabolic partial differential equations with Volterra nonlinearities are given in Vazquez and Krstic (2008a,b). In particular, these results are applicable to semilinear parabolic equations. The constructed control laws are expressed by Volterra series. The authors prove the *local* exponential stability.

In Vazquez and Krstic (2008a,b), a feedback law is proposed to locally stabilize stationary profiles for arbitrarily large reaction coefficients and lengths of the system. Here we show that, under restrictions on the magnitude of the reaction term and the length of the system, a simpler feedback law using boundary values only can globally exponentially stabilize any reference trajectory.

In this paper, a 1-d system of length L is studied. In the reaction–diffusion equation, the diffusion coefficient is normalized to 1. The parameter K determines the size of the reaction term. To show the exponential decay of the solution, it is assumed that L^2K is sufficiently small. Thus, if the reaction rate K is large, the space interval $[0, L]$ has to be sufficiently short. In this case, Lemma 1 states that the stationary states are uniquely determined by the corresponding boundary value problems. An example illustrates that, if $L^2 K$ is too large, several stationary states may exist that satisfy the same Robin boundary conditions. Thus, in this situation it is impossible to stabilize the system using these Robin boundary conditions.

This paper has the following structure: In Section 2, the model is defined and a result about the well-posedness is given. Moreover, the stationary states and time-dependent orbits are discussed. In Section 3, the result about two-sided boundary feedback stabilization is presented: if the length L of the reactor is sufficiently small, there is a feedback constant $C > 0$ such that the Robin boundary conditions ensure stability. Numerical experiments illustrate the results. Section 4 contains conclusions.

2. The model

2.1. Definition of the model

In this section, the Schlögl model is defined.

Let real numbers $u_1 \leq u_2 \leq u_3$ be given. Define the polynomial

$$\varphi(u) = (u - u_1)(u - u_2)(u - u_3). \tag{1}$$

Due to its definition, φ has the property

$$m_\varphi = \inf_{u\in(-\infty,\infty)} \varphi'(u) > -\infty, \tag{2}$$

that is the derivative of φ is bounded below. The infimum $m_\varphi < 0$ is attained at the point $(u_1 + u_2 + u_3)/3$.

The system that is considered in this paper is governed by the semilinear parabolic partial differential equation

$$u_t = u_{xx} - K\varphi(u) \tag{3}$$

with a constant $K > 0$ complemented by appropriate initial and boundary conditions. In the reaction–diffusion equation (3), the diffusion coefficient is equal to 1 and the constant K determines the size of the reaction term. If K equals zero, the reaction term vanishes and the partial differential equation (3) models a pure diffusion process.

Let the length $L > 0$ be given. Let $u^{stat} \in H^2(0, L)$ denote a stationary solution of (3), that is $u = u^{stat}$ solves the equation

$$u_{xx}(x) = K\varphi(u(x)), \quad x \in [0, L]. \tag{4}$$

To define a feedback law, introduce a real constant $C > 0$. For the stabilization of (3), for $(t, x) \in [0,\infty) \times [0, L]$, consider the Robin boundary conditions

$$u_x(t, 0) = C(u(t, 0) - u^{stat}(0)) + u_x^{stat}(0), \tag{5}$$

$$u_x(t, L) = -C(u(t, L) - u^{stat}(L)) + u_x^{stat}(L). \tag{6}$$

Notice that the boundary values $u_x^{stat}(0)$ and $u_x^{stat}(L)$ are well defined since $u_x^{stat} \in H^1(0, L)$.

With the feedback laws (5), (6), if

$$L^2 K < \frac{1}{2\,|m_\varphi|},$$

the Lyapunov function presented in Theorem 3 decays exponentially.

2.2. Existence and uniqueness of the solutions

In Buchholz et al. (2013), the well-posedness of the system governed by (3) is studied for homogeneous Neumann boundary conditions. It is shown that for initial data in $L^\infty(0, L)$, the system has a unique weak solution that is continuous for $t > 0$. If the initial state is continuous, the solution of the system is continuous for all times. The same result extends to the Robin boundary conditions (5), (6). In the associated theorem below, the standard Sobolev space

$$W(0, T) = L^2(0, T, H^1(0, L)) \cap H^1(0, T; H^1(0, L)')$$

is used. Moreover, the notation

$$Q_T = (0, T) \times (0, L)$$

is used.

Theorem 1. *Suppose that it holds $K \geq 0$ and $u_1 < u_2 < u_3$. Then, for all $f \in L^2(Q_T)$, $u_0 \in L^\infty(0, L)$, $g_i \in L^p(0, T)$, $i = 1, 2$, $p > 2$, the*