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Brief paper

Finite-time and fixed-time stabilization: Implicit Lyapunov function approach☆

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ABSTRACT

Theorems on Implicit Lyapunov Functions (ILF) for finite-time and fixed-time stability analysis of nonlinear systems are presented. Based on these results, new nonlinear control laws are designed for robust stabilization of a chain of integrators. High order sliding mode (HOSM) algorithms are obtained as particular cases. Some aspects of digital implementations of the presented algorithms are studied, it is shown that they possess a chattering reduction ability. Theoretical results are supported by numerical simulations.

1. Introduction and related works

Many practical applications require severe time response constraints (for security reasons, or simply to improve productivity). That is why, finite-time stability and stabilization problems have been intensively studied, see (Bhat & Bernstein, 2000; Haimo, 1986; Moulay & Perruquetti, 2006; Orlov, 2009; Roxin, 1966). Time constraint may also appear in observation problems when a finite-time convergence of the state estimate to the real values is required: Bejarano and Fridman (2010), Engel and Kreisselmeier (2002), Menard, Moulay, and Perruquetti (2010), Perruquetti, Floquet, and Moulay (2008), Shen, Huang, and Gu (2011).

Let us stress that finite-time stability property is frequently associated with HOSM controls, since these algorithms should ensure finite-time convergence to a sliding manifold (Levant, 2005a; Orlov, 2005; Polyakov & Poznyak, 2009; Utkin, Guldner, & Shi, 2009). Typically, the associated controllers have mechanical and electromechanical applications (Bartolini, Pisano, Punta, & Usai, 2003; Chernousko, Anan'evskii, & Reshmin, 2008; Dinuzzo & Ferrara, 2009; Ferrara & Giacomini, 1998).

The theoretical background of HOSM control systems is very well developed (Levant, 2005a,b, 2007). However, applications of the existing HOSM control algorithms are complicated, since there are a few constructive algorithms for tuning the HOSM control parameters. Most of them are restricted to the second order sliding mode systems (Cruz-Zavala, Moreno, & Fridman, 2011; Levant, 2007; Polyakov, 2012; Polyakov & Poznyak, 2009).

Fixed-time stability, that demands *boundedness of the settling-time function* for a globally finite-time stable system, was studied in Cruz-Zavala et al. (2011), Polyakov (2012), Polyakov and Fridman (2014). This property was originally discovered in the context of homogeneous systems (Andrieu, Praly, & Astolfi, 2008). Fixed-time stability looks promising if a controller (observer) has to be designed in order to provide some required control (observation) precision in *a given time* and *independently of initial conditions*.

The main tool for analysis of finite-time and fixed-time stability is the Lyapunov function method (see, for example, Bhat & Bernstein, 2000; Moulay & Perruquetti, 2006; Polyakov & Fridman, 2014), which is lacking for constructive design in the nonlinear case. This paper deals with ILF method (Adamy & Flemming, 2004), which relies on Lyapunov functions defined, implicitly, as solutions to an algebraic equation. Stability analysis does not require solving this equation, since the implicit function theorem (Courant & John, 2000) helps to check all stability conditions directly from the implicit formulation. The similar approach was presented in Korobov (1979) for control design and called the controllability function method (see, also Anan'evskii, 2011).

This paper addresses the problem of a control design for the robust finite-time and fixed-time stabilization of a chain of

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integrators. The ILF method is used to design the control laws together with Lyapunov functions for closed-loop systems. This method allows us to analyze robustness of the closed-loop system and to design *a high order sliding mode control algorithm*, which rejects bounded matched exogenous disturbances. Finite-time and fixed-time stability conditions were obtained in the form of Linear Matrix Inequalities (LMI). They provide *simple constructive schemes for tuning the control parameters* in order to predefine the required convergence time and/or to guarantee stability and robustness with respect to disturbances of a given magnitude.

Through the paper the following notations will be used:

- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$;
- $\frac{dV}{dt}\big|_{(.)}$ is the time derivative of a function V along the solution of a differential equation numbered as (.);
- $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$;
- $\mathrm{diag}\{\lambda_i\}_{i=1}^n$ is the diagonal matrix with the elements λ_i;
- a continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is monotone increasing and $\sigma(h) \to 0^+$ as $h \to 0^+$;
- for a symmetric matrix $P = P^T$ the minimal and maximal eigenvalues are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$,
- $\mathrm{int}(\Omega)$ is the interior of the set $\Omega \subseteq \mathbb{R}^n$.

2. Problem statement

The paper deals with finite-time and fixed-time stabilization problems for the disturbed chain of integrators. Note that a control design scheme developed for such systems usually admits extension to feedback linearizable nonlinear multi-input multi-output systems (Isidori, 1995). The problem statement presented below is also typical for high-order sliding mode control design (Levant, 2005a).

Consider a linear single input system of the following form:

$$\dot{x}(t) = Ax(t) + bu(t) + d(t, x(t)), \quad t > 0, \tag{1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix},$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, and the function $d : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes the system uncertainties and disturbances. The whole state vector x is assumed to be measured. Let the function d be measurable locally bounded uniformly in time, i.e. $\sup_{t\in\mathbb{R}_+, x\in\mathbb{R}^n:\|x\|<\delta} \|d(t, x)\| < \infty$ for any $\delta > 0$. Both the control function u and the function d are admitted to be discontinuous with respect to x. For example, the function d may describe unknown dry friction of a mechanical model. The analysis of such systems requires a special mathematical framework. In this paper we use Filippov theory (Filippov, 1988).

The goal of the paper is to develop control laws such that the origin of the closed-loop system (1) will be globally asymptotically stable and all its trajectories will reach the origin in a finite time or in the fixed time $T_{\max} \in \mathbb{R}_+$. In addition, the control algorithms to be developed should have effective schemes for tuning the control parameters and assigning of the settling time.

The control design is based on ILF approach to finite-time and fixed-time stability analysis, which is developed in the next section.

3. Preliminaries

3.1. Finite-time and fixed-time stability

Consider the system defined by

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field locally bounded uniformly in time. If f is a locally measurable function that is discontinuous with respect to the state variable x, then a solution of the Cauchy problem (2) is understood in the sense of Filippov (1988), namely, as an absolutely continuous function satisfying the differential inclusion

$$\dot{x}(t) \in K[f](t, x(t))$$

for almost all $t \in [0, t^*)$, where $t^* \in \mathbb{R}_+$ or $t^* = +\infty$. The set-valued function $K[f] : \mathbb{R}_+ \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined for any fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ as follows:

$$K[f](t, x) = \bigcap_{\varepsilon>0} \bigcap_{N:m(N)=0} \mathrm{co} f(t, B(x, \varepsilon) \setminus N),$$

where $\mathrm{co}(M)$ defines the convex closure of the set $M \subset \mathbb{R}^n$, $B(x, \varepsilon)$ is the ball with the center at $x \in \mathbb{R}^n$ and the radius $\varepsilon \in \mathbb{R}_+$, the equality $m(N) = 0$ means that the Lebesgue measure of the set $N \subset \mathbb{R}^n$ is zero.

Let the origin be an equilibrium of (2), i.e. $0 \in K[f](t, 0)$. The system (2) may have non-unique solutions and may admit both weak and strong stability (see, for example, Filippov, 1988). This paper deals only with the *strong stability* properties, which ask for stable behavior of all solutions of the system (2). The next definition of uniform finite-time stability is just a compact representation of Definition 2.5 from Orlov (2005).

Definition 1. The origin of system (2) is said to be globally **uniformly finite-time stable** if it is globally uniformly asymptotically stable (see, for example, Orlov, 2005) and there exists a **locally bounded** function $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, such that $x(t, x_0) = 0$ for all $t \geq T(x_0)$, where $x(\cdot, x_0)$ is an arbitrary solution of the Cauchy problem (2). The function T is called the **settling-time function**.

Asymptotic stability of the time-independent (autonomous) system always implies its uniform asymptotic stability (see, for example, Clarke, Ledyaev, & Stern, 1998). For finite-time stable systems this is not true in general case (see, for example, Polyakov & Fridman, 2014), since Definition 1 additionally asks a uniformity of the settling time with respect to initial conditions.

The origin of system $\dot{x}(t) = -|x(t)|^{0.5}\mathrm{sign}[x(t)]$, $x \in \mathbb{R}$ is globally uniformly finite-time stable, since its settling-time function T is locally bounded: $T(x_0) = 2\sqrt{|x_0|}$.

Definition 2 (*Polyakov, 2012*)**.** The origin of system (2) is said to be globally **fixed-time stable** if it is globally uniformly finite-time stable and the settling-time function T is **globally bounded**, i.e. $\exists T_{\max} \in \mathbb{R}_+$ such that $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

The presented definition just asks more: strong uniformity of finite-time stability with respect to initial condition. The origin of system $\dot{x}(t) = -\left(|x|^{0.5}(t) + |x|^{1.5}(t)\right)\mathrm{sign}(x(t))$, $x \in \mathbb{R}$, is globally fixed-time stable, since its settling time function $T(x_0) = 2\arctan(\sqrt{|x_0|})$ is bounded by $\pi \approx 3.14$.

The uniformity of finite-time and fixed-time stability with respect to system disturbances can also be analyzed. For instance, finite-time stability, which is uniform (in some sense) with respect to both initial conditions and system disturbances, was called equiuniform finite-time stability (Orlov, 2005).

3.2. Implicit Lyapunov function method

This subsection introduces some stability theorems further used for control design. They refine the known results about global uniform asymptotic, finite-time and fixed-time stability of differential inclusions to the case of implicit definition of Lyapunov function. The next theorem extends Theorem 2 from Adamy and Flemming (2004).

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