Automatica 51 (2015) 341–347

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Performance and robustness analysis of stochastic jump linear systems using Wasserstein metric☆

Kooktae Lee [a], Abhishek Halder [b], Raktim Bhattacharya [a]

[a] Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843-3141, USA

[b] Department of Electrical & Computer Engineering, Texas A&M University, College Station, TX 77843-3141, USA

ARTICLE INFO

Article history:
Received 5 December 2013
Received in revised form
18 August 2014
Accepted 8 September 2014
Available online 6 November 2014

Keywords:
Performance and robustness analysis
Stochastic jump linear systems
Switched linear systems
Wasserstein distance

ABSTRACT

This paper focuses on the performance and robustness analysis of stochastic jump linear systems. In the presence of stochastic jumps, state variables evolve as random process, with associated time varying probability density functions. Consequently, system analysis is performed at the density level and a proper metric is necessary to quantify the system performance. In this paper, Wasserstein metric that measures a distance between probability density functions is employed to develop new results for the performance analysis of stochastic jump linear systems. Both transient and steady-state performance of the systems, with given initial state uncertainties, can be analyzed in this framework. Also, we prove that the convergence of the Wasserstein metric implies the mean square stability. We present a novel "Split-and-Merge" algorithm for propagation of state uncertainty in such systems. Overall, this study provides a unifying framework for the performance and robustness analysis of general stochastic jump linear systems, and not necessarily Markovian that is commonly assumed. The usefulness and efficiency of the proposed method are verified through numerical examples.

1. Introduction

A jump linear system is defined as a dynamical system constructed with a set of linear subsystem dynamics and a switching logic that conduct a switching between linear subsystems. Over decades, a jump linear system has attracted a wide range of researches due to its practical implementations. For instance, jump linear systems are used for power systems, manufacturing systems, aerospace systems, networked control systems, etc.

In general, a jump linear system can be divided into two different categories depending on the switching logic. One branch is a *deterministic jump linear system* where the jump sequence is driven deterministically by a certain switching logic. The utilization of such systems stems from plant stabilization (Minto & Ravi, 1991), adaptive control (Narendra & Balakrishnan, 1994), system performance (Lin & Antsaklis, 2009), and resource-constrained scheduling (Boctor, 1990). In most cases, the system stability has been one of the major issues to investigate since even stable subsystems make the system unstable by the switching. Hence, numerous results have been established for the stability of such systems, and the recent literature (Lin & Antsaklis, 2009) provides the necessary and sufficient conditions for asymptotic stability.

Unlike the deterministic jump linear system, a *stochastic jump linear system* (SJLS), which is another category of jump linear systems, refers to systems with a stochastic switching process. This type of jump linear systems is commonly used to represent the randomness in the switching such as communication delays or packet losses in the networked control systems (Hassibi, Boyd, & How, 1999; Xiao, Hassibi, & How, 2000). In Hassibi et al. (1999), the networked control system with packet losses was modeled as an asynchronous dynamical system incorporating both discrete and continuous dynamics, and its stability was analyzed through Lyapunov techniques. Since then, this problem has been formulated in a more general setting by representing the various aspects of communication uncertainties as Markov chains (Coviello, Minero, & Franceschetti, 2011; Liu, Ho, & Niu, 2009; Xiong & Lam, 2007; You & Xie, 2011). Stability analysis in the presence of such uncertainty, has been performed in the Markov jump linear systems (MJLSs) framework (Karan, Shi, & Kaya, 2006; Lee & Dullerud,

☆ This research was supported by National Science Foundation award #1349100, with Dr. Almadena Y. Chtchelkanova as a program manager. The material in this paper was partially presented at the 2014 American Control Conference (ACC 2014), June 4–6, 2014, Portland, Oregon, USA. This paper was recommended for publication in revised form by Associate Editor Fabrizio Dabbene under the direction of Editor Roberto Tempo.

E-mail addresses: animodor@tamu.edu (K. Lee), ahalder@tamu.edu (A. Halder), raktim@tamu.edu (R. Bhattacharya).

http://dx.doi.org/10.1016/j.automatica.2014.10.080
0005-1098/

2006; Xiao et al., 2000; Zhang, Shi, Chen, & Huang, 2005; Zhang & Boukas, 2009). Especially, Zhang and Boukas (2009) analyzed the stability of MJLS without requiring any knowledge of the structures in partially unknown Markov transition probabilities. Further, the stochastic stability for a class of nonlinear stochastic systems with semi-Markovian jump parameters is introduced in Hou, Luo, Shi, and Nguang (2006) and Li, Wu, and Shi (2013). Most previous literatures, however, have only dealt with steady-state analysis in terms of the system stability.

Beyond the current literature, this paper has a key contribution for the analysis of a SJLS as follows. Based on the theory of optimal transport (Villani, 2008), we propose new probabilistic tools for analyzing the performance and robustness of SJLSs. Compared to the current literatures that only guarantee asymptotic performance with a deterministic arbitrary initial state condition, our contribution is to develop a unifying framework enabling both transient and asymptotic performance analysis with uncertain initial state conditions. The main difficulty dealing with analysis of SJLSs is that the system trajectories differ from every run due to the random switching. Moreover, the system state for SJLSs becomes random variables with a corresponding probability density function (PDF) even with a deterministic initial state condition. Therefore, we need to adopt a proper metric to measure the performance and robustness of SJLSs in the distributional sense. In this paper, the Wasserstein metric that enables uncertainty quantification by evaluating a distance between PDFs is employed to measure the performance of SJLSs. We also prove that the convergence of this metric implies the mean square stability. To sum up, this paper provides a *new framework* for the performance and robustness analysis of SJLSs in the existence of initial state uncertainties and without any restriction on the underlying jump processes.

The remainder of this paper is organized as follows. In Section 2, we provide a brief review of the preliminaries. Section 3 deals with the performance and robustness analysis of SJLSs, and we propose a computationally efficient tool for this purpose. Numerical examples are provided in Section 4, to illustrate results for the performance analysis, developed in this work. Section 5 concludes the paper.

Notation: The set of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Further, $\mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$. The symbols $\mathrm{tr}(\cdot)$, $\otimes$, and $vec(\cdot)$ denote the trace of a square matrix, Kronecker product, and vectorization operators, respectively. The abbreviation m.s. stands for the convergence in mean-square sense. The notations $\mathbb{P}(\cdot)$ and $X \sim \rho(x)$ represent the probability and the random variable X with PDF $\rho(x)$, respectively. The symbol $\mathcal{N}(\mu, \Sigma)$ is used to denote the PDF of a Gaussian random vector with a mean μ and a covariance Σ.

2. Preliminaries

Consider a discrete-time jump linear system as follows.

$$x(k+1) = A_{\sigma_k} x(k), \quad k \in \mathbb{N}_0 \tag{1}$$

where k is a discrete-time index, $x(k)$ is the state vector, and A_{σ_k} denotes the system matrices. $\sigma_k \in \mathcal{M} \triangleq \{1, 2, \ldots, m\}$ stands for the stochastic jump process, representing the switching among m different modes of (1). In this paper, we will consider general stochastic jump processes σ_k, and hence σ_k can be any arbitrary random process. Then, the resulting dynamics becomes a SJLS as defined next.

Definition 1 (*Stochastic Jump Linear System*)**.** Tuple of the form $(\{\pi(k)\}_{k=1}^{\infty}, \{A_1, \ldots, A_m\})$ is termed as a SJLS, provided the mode dynamics are given by (1); $\pi(k)$ denotes the occupation probability vector at time k for the prescribed stochastic process σ_k.

Remark 1. A SJLS, as defined above, consists of a sequence of mode-occupation probability vectors and a set of subsystem dynamics. If the jump processes σ_k is deterministic, then at each time, $\pi(k)$ will have integral co-ordinates (single 1 and remaining $m-1$ zeros), resulting in a *deterministic switching sequence*. If, however, σ_k is stochastic jump processes, then $\pi(k)$ will contain proper fractional co-ordinates, resulting in a *randomized switching sequence* where at each time, exactly one out of m modes will be chosen according to probability $\pi(k)$. Thus, starting from a deterministic initial condition, each execution of the SJLS may lead to different switching sequences σ_k, and hence results in realization of different trajectories on the state space. As a consequence, even with a fixed initial condition, repeated executions of stochastic jumps yield a spatio-temporal evolution of joint state PDF $\rho(x(k))$.

Next, we distinguish several different categories of SJLSs according to inherent stochastic jump processes as follows.

(1) i.i.d. jump process:
A SJLS switching probability is called stationary, if the occupation probability vector $\pi(k)$ remains stationary in time. In particular, a stationary *deterministic* switching sequence implies execution of a single mode (no switching). A stationary randomized switching sequence implies i.i.d. jump process.

(2) Markov jump process:
Consider a discrete-time discrete state Markov chain with mode transition probabilities given by

$$p_{ij} = \mathbb{P}(\sigma_{k+1} = j \mid \sigma_k = i)$$

where $p_{ij} \geq 0, \forall i, j \in \mathcal{M}$. Hence, for $k \geq 0$, the switching probability $\pi(k) \in \mathbb{R}^m$ of the modes of (1), is governed by

$$\pi(k+1) = \pi(k)P, \quad \pi(0) = [\pi_1(0) \; \cdots \; \pi_m(0)]$$

where $\pi(0)$ is a given initial switching probability. The *Markov transition probability matrix* $P \in \mathbb{R}^{m\times m}$ is a right stochastic matrix with row sum $\sum_{j=1}^{m} p_{ij} = 1, \forall i \in \mathcal{M}$.

(3) Semi-Markov jump process:
For a homogeneous and discrete-time semi-Markov chain, semi-Markov kernel q is defined by

$$q_{ij}(k) = \mathbb{P}(\sigma_{n+1} = j, X_{n+1} = k \mid \sigma_n = i)$$

where X_n denotes the sojourn time in state $\sigma_n = i$. Note that the transition probability p_{ij} in Markov chain can be expressed in terms of the semi-Markov kernel by $p_{ij} = \sum_{k=0}^{\infty} q_{ij}(k)$.

3. Performance and robustness analysis using Wasserstein metric

Uncertainties in a SJLS appear at execution level due to random switching sequence. Additional uncertainties may stem from imprecise setting of initial conditions and parameter values. These uncertainties manifest as the evolution of the state PDF $\rho(x(k))$. Thus, a natural way to quantify the uncertainty for the performance of a SJLS, is to compute the "distance" of the instantaneous state PDF from a reference measure. In particular, if we fix the reference PDF as Dirac delta function at the origin, denoted as $\delta(x)$, then the time–history of this "distance" would reveal the rate of convergence (divergence) for the stable (unstable) SJLS in the distributional sense. For meaningful inference, the notion of "distance" must define a metric, and should be computationally tractable. For this purpose, we adopt the Wasserstein distance and details are introduced in the following subsection.