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Brief paper

Synthesis of Razumikhin and Lyapunov–Krasovskii approaches to stability analysis of time-delay systems ☆

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ABSTRACT

In this paper, a necessary and sufficient condition for the exponential stability of linear systems with several time-delays is presented. Such a condition is based on the construction of quadratic lower bounds for the Lyapunov–Krasovskii functionals on the special Razumikhin-type set of functions. The result reveals a constructive procedure for the stability analysis whose application is illustrated with examples.

1. Introduction

In this paper, we analyze the stability of linear time-invariant differential systems with several delays within the framework of the Lyapunov–Krasovskii functionals. The basic concepts of the Lyapunov–Krasovskii approach and, in particular, the structure of the functionals used, were first established in Repin (1965), Krasovskii (1956) and further developed in Huang (1989), Infante and Castelan (1978) and Kharitonov and Zhabko (2003). More precisely, in Kharitonov and Zhabko (2003) the so-called complete-type functionals were introduced. These functionals admit quadratic lower and upper bounds, and for this reason, at the present time, they are effectively used for construction of exponential estimates of solutions (Kharitonov, 2013), for the stability (Egorov & Mondie, 2013) and the robust stability analysis with respect to uncertainties in coefficients (Kharitonov, 2013) or in delay (Kharitonov & Niculescu, 2003), for computation of critical delays (Ochoa, Mondie, & Kharitonov, 2009). The book of Kharitonov (2013) gives a detailed survey on the present state of the art in the area.

A different modification of the approach is considered in this paper. Following the work of Huang (1989), we use a functional with time derivative given as a negative definite quadratic form of the present state of the system. To apply this functional to stability analysis, one needs to obtain a quadratic lower bound on it. However, in Huang (1989) this functional is shown to admit only a local cubic lower bound on the set of solutions of the system. As a result, it is considered not to be effective in the stability analysis and applications.

On the contrary, in this contribution we propose a constructive method for stability analysis using the functional studied in Huang (1989). The idea of the method is to use a special Razumikhin-type condition (see Razumikhin, 1956) for the estimation of the functional. It turns out that for the stability analysis it is sufficient to construct a lower bound for the functional on the set of functions satisfying this condition instead of the set of solutions. This enables us to establish a necessary and sufficient condition for the exponential stability, and opens a constructive way for analysis. The proposed method provides possibility to compute the critical delays (see Medvedeva & Zhabko, 2013) and to perform the robustness analysis, as well.

The paper is organized as follows. After some preliminary definitions in Section 2, we present a constructive necessary and sufficient condition for the exponential stability (Theorem 1) in Section 3. Then, Section 4 provides a detailed exposition of the method for stability analysis which is based on the theoretical results of Section 3. In Section 5, we discuss the convergence issue (see Theorem 10) which plays a key role in the application of the method. In Section 6, we give the illustrative examples. Some concluding remarks end the paper.

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2. Preliminaries

We consider a time-delay system of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t - h_j), \tag{1}$$

where $A_j \in \mathbb{R}^{n\times n}, j = 0, 1, \dots, m$, are given constant matrices, and $0 = h_0 < h_1 < \cdots < h_m = h$ are the delays. Let $t_0 = 0$ be an initial time instant, and $\varphi(\theta)$ be an initial function which is supposed to be piecewise continuous on the segment $[-h, 0]$. We denote the solution of system (1) with the initial function φ by $x(t, \varphi)$, and the segment of the trajectory $\{x(t+\theta, \varphi) \mid \theta \in [-h, 0]\}$ by $x_t(\varphi)$. We omit the symbol φ when no confusion can arise. Throughout the paper, we use the Euclidean norm for vectors, and the analogue of the uniform norm for vector functions $\|\varphi\|_h = \sup_{\theta\in[-h,0]} \|\varphi(\theta)\|$.

It is well-known (see Bellman & Cooke, 1963) that the characteristic equation of system (1) is of the form $\det\left(\lambda I - \sum_{j=0}^{m} A_j e^{-\lambda h_j}\right) = 0$, and its roots are called eigenvalues. System (1) is said to satisfy the Lyapunov condition if it does not have the eigenvalues whose sum is equal to zero.

Given a symmetric matrix W, we say that $U(\tau)$ is the Lyapunov matrix associated with W if it satisfies the following set of equations (see Kharitonov, 2013)

$$\begin{aligned} &U'(\tau) = \sum_{j=0}^{m} U(\tau - h_j)A_j, \quad \tau \geqslant 0; \\ &U(-\tau) = U^T(\tau), \quad \tau \geqslant 0; \\ &\sum_{j=0}^{m} [U(-h_j)A_j + A_j^T U^T(-h_j)] = -W. \end{aligned} \tag{2}$$

The Lyapunov matrix exists and is unique for any symmetric matrix W, if and only if system (1) satisfies the Lyapunov condition, see Kharitonov (2013). According to Kharitonov and Zhabko (2003), the matrix $U(\tau)$ associated with W determines the quadratic functional

$$\begin{aligned} v(x_t) &= x^T(t)U(0)x(t) \\ &+ 2x^T(t) \sum_{j=1}^{m} \int_{-h_j}^{0} U(-\theta - h_j)A_j x(t+\theta)d\theta \\ &+ \sum_{k=1}^{m}\sum_{j=1}^{m} \int_{-h_k}^{0} x^T(t+\theta_1)A_k^T \\ &\times \left(\int_{-h_j}^{0} U(\theta_1 + h_k - \theta_2 - h_j)A_j x(t+\theta_2)d\theta_2\right)d\theta_1, \end{aligned} \tag{3}$$

such that its time derivative along the solutions of system (1) is equal to $-x^T(t)Wx(t)$. One can check the last fact directly, using properties (2) of the Lyapunov matrix.

To define a critical delay, we first suppose that $h_j = \alpha_j \mathfrak{h}$, $j = 1, 2, \dots, m$, where $\alpha_j > 0$ are fixed, and $\mathfrak{h} \geqslant 0$ is a basic delay. The values of basic delay for which system (1) changes the property of exponential stability to the instability, or vice versa, are said to be the critical delays.

Finally, let us introduce the set

$$S = \left\{\varphi : \|\varphi(\theta)\| \leqslant \|\varphi(0)\|,\ \theta \in [-h, 0]\right\}.$$

The set S is the very set whereon we will further check the positive definiteness of the functional v.

3. Stability theorem

Here we present our main stability result.

Theorem 1. *Given a positive definite matrix W, system (1) is exponentially stable, if and only if there exists a functional $v(\varphi)$ such that the following conditions hold:*

1. $\dfrac{dv(x_t)}{dt} = -x^T(t)Wx(t)$ *along the solutions of system (1);*
2. *On the set S the functional admits a lower bound of the form*

$$v(\varphi) \geqslant \mu\|\varphi(0)\|^2, \quad \mu > 0.$$

Proof. Necessity. The idea of the proof of the necessity part is borrowed from Huang (1989). Let v be a functional of the form (3), then the first condition of the theorem holds, as was mentioned in the previous section. To prove the second one, take an arbitrary function $\varphi \in S$, and set $\alpha = \|\varphi\|_h$. Since $\varphi \in S$, we have $\|\varphi(0)\| = \alpha$. Integrating system (1) and using Gronwall's lemma (see Bellman & Cooke, 1963, p. 31), we obtain that

$$\|x(t, \varphi)\| \leqslant N(t), \quad \text{where } N(t) = \alpha K_1 e^{Kt},$$
$$K = \sum_{j=0}^{m} \|A_j\|, \qquad K_1 = 1 + \sum_{j=1}^{m} \|A_j\|h_j.$$

Hence, $\|\dot{x}(t, \varphi)\| \leqslant KN(t) \leqslant KN(\delta)\ \forall t \leqslant \delta$, $\forall\delta > 0$, and $\|x(t, \varphi) - \varphi(0)\| \leqslant KN(\delta)\delta$, $t \leqslant \delta$. Let us choose δ so that $KN(\delta) = \alpha/(2\delta)$, note that δ does not depend on α. Then,

$$\|x(t, \varphi)\| \geqslant \|\varphi(0)\| - KN(\delta)\delta = \|\varphi(0)\|/2, \quad t \leqslant \delta.$$

Since system (1) is exponentially stable, we have

$$\begin{aligned} v(\varphi) &= \int_0^\infty x^T(t, \varphi)Wx(t, \varphi)dt \\ &\geqslant \lambda_{\min}(W)\int_0^\delta \|x(t, \varphi)\|^2 dt \geqslant \lambda_{\min}(W)\delta\frac{\|\varphi(0)\|^2}{4}, \end{aligned}$$

where $\lambda_{\min}(W)$ is the smallest eigenvalue of W. Thus, $\mu = \lambda_{\min}(W)\delta/4 > 0$, and the proof of necessity is complete. Note that μ is obtained constructively.

Sufficiency. Suppose that there exists a functional of the form (3) satisfying the second condition of the theorem but system (1) is not exponentially stable. Then there exists a sequence $\{t_k\}_{k=1}^{\infty}$, such that $t_k - t_{k-1} \geqslant h$, $t_k \xrightarrow[k\to+\infty]{} +\infty$, and $\|x(t_k)\| \geqslant \beta > 0$. Consider two cases.

1. Let the solution $x(t)$ be uniformly bounded, i.e. there exists $G > 0$ such that $\|x(t)\| \leqslant G\ \forall t \geqslant 0$. Then, $\|\dot{x}(t)\| \leqslant KG\ \forall t \geqslant 0$, where $K = \sum_{j=0}^{m} \|A_j\|$.

Take $t \in [t_k, t_k + \tau]$, $\tau > 0$. Then, $\|x(t) - x(t_k)\| \leqslant KG(t - t_k) \leqslant KG\tau$, and, choosing $\tau = \min\left\{\frac{\beta}{2KG}; h\right\}$, we obtain

$$\|x(t)\| \geqslant \|x(t_k)\| - KG\tau \geqslant \frac{\beta}{2}, \quad t \in [t_k, t_k + \tau],$$

for every k. Further, let $N(t)$ be the number of intervals $[t_k, t_k + \tau]$, contained in $[0, t]$. These intervals do not intersect with each other due to the choice of τ, and $N(t) \xrightarrow[t\to+\infty]{} +\infty$. Hence,

$$\begin{aligned} \int_0^t x^T(s)Wx(s)ds &\geqslant \sum_{k=1}^{N(t)} \int_{t_k}^{t_k+\tau} x^T(s)Wx(s)ds \\ &\geqslant \lambda_{\min}(W)\frac{\beta^2\tau}{4}N(t) \xrightarrow[t\to+\infty]{} +\infty. \end{aligned}$$