Automatica 51 (2015) 175–179

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Cycles of periodically time-variant Boolean networks[☆]

Yunlei Zou, Jiandong Zhu[1]

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, PR China

ARTICLE INFO

Article history:
Received 21 May 2013
Received in revised form
7 June 2014
Accepted 3 October 2014

Keywords:
Periodically time-variant Boolean network
Cycle
Semi-tensor product

ABSTRACT

This paper investigates cycles of periodically time-variant Boolean networks (PTVBNs). Some properties of cycles of PTVBNs and the relationship between PTVBNs and time-invariant Boolean networks (TIBNs) are revealed. It is shown that each cycle of a PTVBN corresponds to a common cycle of some TIBNs. And, all the cycles of a PTVBN can be constructed by those of the corresponding TIBNs. Moreover, some results on lengths of cycles of PTVBNs are derived. Finally, some examples are given to illustrate the obtained theoretical results.

1. Introduction and preliminaries

Boolean network (BN) theory was first proposed by Kauffman (1969) to describe cellular networks and genetic regulatory networks. In modeling of genetic regulatory networks, cycles are often associated with the behavior of cells (see Akutsu, Miyano, & Kuhara, 1999; Aracena, Gonzalez, Zuniga, Mendez, & Cambiazo, 2006; Huang, 1999; Kauffman, Peterson, Samuelsson, & Troein, 2003 and Shmulevich, Dougherty, & Zhang, 2002). For Boolean networks (BNs), one of the most important topics is to find all the cycles. In recent years the semi-tensor product of matrices, proposed by Cheng, has been used to analyze BNs (Cheng, Qi, & Li, 2011). It is a new technique that can convert logical dynamic equations of BNs into discrete-time dynamic systems. With this new technique, the topological structures of time-invariant Boolean networks (TIBNs) can be revealed completely (Cheng & Qi, 2010). Furthermore, many classical problems of control theory have been extended to Boolean control networks (BCNs) such as controllability, observability (Cheng & Qi, 2009), stabilization (Cheng, Qi, Li, & Liu, 2011), Kalman decomposition (Cheng, Li, & Qi, 2010), disturbance decoupling (Cheng, 2011) and optimal control (Zhao, Li, & Cheng, 2011). But for time-variant Boolean networks (TVBNs), there are only a few results reported. In Li and Wang (2012), reachability and controllability are generalized to switched BNs, which are actually a special kind of time-variant systems. In Zhang and Zhang (2013), for general time-variant BCNs, a necessary and sufficient condition for the controllability is obtained and a control design algorithm is presented. For a kind of temporal BNs with time-variant delays, controllability, optimal control and synchronization problems are investigated (Li & Lu, 2013; Li & Sun, 2012).

It is well-known that, for the traditional dynamical systems, time-variant systems have many essential properties different from time-invariant systems. Specially, periodically time-variant systems have some particular interesting dynamical behaviors. And they exist widely in BNs such as the genetic regulatory networks with periodical external inputs to the genome (Ballesteros & Luque, 2002), the switched BNs (Li & Wang, 2012) with periodical switching signals, the perturbed BNs with periodical function perturbations (Xiao & Dougherty, 2007) and a class of BCNs with dynamical controllers (Cheng, 2009). In the real world, many biological rhythms have a 24-h period related to sunlight and many genetic regulatory networks are influenced by some periodic medical interventions. In different environments, the modeled BNs are different. The physical meaning of PTVBNs just lies in the periodical model transition among different BNs. A natural problem is how the periodical model transition affects the dynamical behavior. This is just the motivation of this paper.

In this paper, we investigate cycles of PTVBNs. Some concepts and basic dynamical properties, which are different from those of TIBNs, are proposed in Section 2. In Section 3, the main results on cycles of PTVBNs are given. The relationship between the cycles of PTVBNs and the common cycles of a group of TIBNs is revealed, with which an effective method for finding cycles is obtained. In

☆ This research was supported in part by National Natural Science Foundation (NNSF) of China under Grant Nos. 11271194, 61273115, 61074115, 10701042 and a Project Funded by Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD). The material in this paper was partially presented at the 32nd Chinese Control Conference (CCC 2013), during July 26–28, 2013, Xi'an, China. This paper was recommended for publication in revised form by Associate Editor Nuno C. Martins under the direction of Editor André L. Tits.

E-mail addresses: zouyl0903@163.com (Y. Zou), jiandongzhu@njnu.edu.cn, zhujd@139.com (J. Zhu).

1 Tel.: +86 13851781823.

http://dx.doi.org/10.1016/j.automatica.2014.10.071
0005-1098/

Section 4, some examples are provided to illustrate the theoretical results. Section 5 is the concluding remarks.

Throughout this paper, we use the following notations.

(1) $\mathrm{Col}(A)$: the set of all columns of matrix A. The ith column of A is denoted by $\mathrm{Col}_i(A)$.
(2) $\Delta_m = \{\delta_m^i | i = 1, 2, \ldots, m\}$, where δ_m^i is the i-th column of $m \times m$ identity matrix I_m.
(3) $\mathscr{R}_{m\times r}$: the set of all the $m \times r$ real matrices. $L \in \mathscr{R}_{m\times r}$ is called a logical matrix if $\mathrm{Col}(L) \subset \Delta_m$. For simplicity, denote $L = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_r}]$ by $\delta_m[i_1, i_2, \ldots, i_r]$. Denote the set of $m \times r$ logical matrices by $\mathscr{L}_{m\times r}$.
(4) $\mathscr{D} = \{\text{True} = 1, \text{False} = 0\}$. Each element in $\mathscr{D}$ is identified with a vector as True $\sim \delta_2^1$ and False $\sim \delta_2^2$. Thus, $\mathscr{D}$ can be regarded as $\Delta_2 = \{\delta_2^1, \delta_2^2\}$.

Definition 1 (*Cheng, Qi, Li, 2011*)**.** Let $A \in \mathscr{R}_{m\times n}$ and $B \in \mathscr{R}_{p\times q}$. Denote by $\alpha = \mathrm{lcm}(n, p)$ the least common multiple of n and p. The left semi-tensor product of A and B is defined as $A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}})$.

2. Cycles of PTVBNs

Consider the PTVBN described by

$$x_i(t+1) = f_i^{\sigma(t)}(x_1(t), \ldots, x_n(t)), \tag{1}$$

where $\sigma(t) = t\%k + 1$ is a periodic function with period k, $x_i \in \mathscr{D}$ a logical variable and f_i^j a logical function for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Regard each x_i as an element of Δ_2 and take the semi-tensor product $x = \ltimes_{i=1}^n x_i \in \Delta_{2^n}$. Then by the procedure given in Cheng, Qi, Li (2011), system (1) can be converted into an algebraic form $x_i(t+1) = M_i^{\sigma(t)} x(t)$, where $M_i^{\sigma(t)} \in \mathscr{L}_{2\times 2^n}$. Multiplying those equations together yields the algebraic form

$$x(t+1) = L_{\sigma(t)} x(t). \tag{2}$$

Conversely, the logical system (1) can be uniquely determined by (2). Let $L_{t+1} = L_{t\%k+1}$ for any $t \geq k$, we can simply write (2) as

$$x(t+1) = L_{t+1} x(t). \tag{3}$$

Definition 2. Consider PTVBN (1). A state $x_0 \in \Delta_{2^n}$ is called a *fixed point*, if $x(t) = x_0$ for any $t \geq 0$. A sequence $\{x(0), x(1), \ldots, x(t), \ldots\}$ determined by (1) is called a *cycle* if there exists an integer $s > 0$ such that $x(t+s) = x(t)$ for any $t \geq 0$. We call sequence $\{x(0), x(1), \ldots, x(t), \ldots\}$ determined by (1) a *cycle with length s*, if

(i) $x(t+s) = x(t)$ for any $t \geq 0$;
(ii) for any $0 < T < s$, there exists $\tilde{t}$ such that $x(\tilde{t} + T) \neq x(\tilde{t})$.

Obviously, a fixed point is a cycle with length 1.

In this paper, we simply denote by $\overline{\{x_1, x_2, \ldots, x_s\}}$ the periodic sequence $S = \{x_1, x_2, \ldots, x_s, x_1, x_2, \ldots, x_s, \ldots\}$. Let

$$C = \overline{\{\tilde{x}(0), \tilde{x}(1), \ldots, \tilde{x}(s-1)\}} \tag{4}$$

be a cycle of (1). By Definition 2, the length of C is less than or equal to s. Moreover, for simplicity, we also call a cycle of (2) as a cycle of $L_{\sigma(t)}$.

Lemma 1. *Consider the PTVBN (1) with algebraic form (2). If $x(mk) = x(0)$ for some non-negative integer m, then $x(mk+l) = x(l)$ for any $l \geq 0$.*

Proof (*Mathematical Induction*)**.** First, as $l = 0$, we have $x(mk+l) = x(l)$. Next, with the assumption $x(mk+l-1) = x(l-1)$, we try to prove $x(mk+l) = x(l)$. Let $r = (l-1)\%k + 1$. Then, by (3) and the induction hypothesis, it follows that $x(mk+l) = L_r x(mk+l-1) = L_r x(l-1) = x(l)$. □

Proposition 1. *Consider the PTVBN (1) with algebraic form (2). The following statements hold:*

(i) *A state $x_0 \in \Delta_{2^n}$ is a fixed point of system (1) if and only if $L_i x_0 = x_0$ for every $i = 1, 2, \ldots, k$.*
(ii) *The periodic sequence C given by (4) is a cycle of PTVBN (1) if and only if*

$$\tilde{x}(t+s) = \tilde{x}(t), \quad \forall\, 0 \leq t \leq ks/(s,k) - s. \tag{5}$$

(iii) *The periodic sequence C given by (4) is a cycle of PTVBN (1) with length s if and only if (5) holds and*

$$\forall\, 0 < T < s,\ \exists 0 \leq \bar{t} \leq s-1, \quad \text{s.t.}\ \tilde{x}(T + \bar{t}) \neq \tilde{x}(\bar{t}). \tag{6}$$

Proof. (i) By (2), this result is easily obtained.

(ii) The necessity is obvious by Definition 2. Here, we prove the sufficiency. Denote the greatest common divisor (s, k) by β and let $k = k_1\beta$, $s = s_1\beta$, where k_1 and s_1 are positive integers. Then $k_1 s = s_1 k$. In the following, we use the Second Principle of Mathematical Induction to prove that $\tilde{x}(t+s) = \tilde{x}(t)$ for any $t \geq 0$. First, condition (5) implies that $\tilde{x}(t+s) = \tilde{x}(t)$ for any $0 \leq t \leq (k_1 - 1)s$. Let $t' > (k_1 - 1)s$ in the following. With the assumption that $\tilde{x}(\tau + s) = \tilde{x}(\tau)$ for all $\tau < t'$, we try to prove $\tilde{x}(t'+s) = \tilde{x}(t')$. Since $t' + s > k_1 s$, we let

$$t' + s = pk_1 s + r = ps_1 k + r, \tag{7}$$

where p and r are non-negative integers, and r satisfies $0 \leq r < k_1 s$. Considering $t' - s < t'$, by the induction hypothesis and (7), we have

$$\tilde{x}(t') = \tilde{x}(t'-s) = \tilde{x}(t'-2s) = \cdots = \tilde{x}(r). \tag{8}$$

Moreover, (5) implies that $\tilde{x}(0) = \tilde{x}(s) = \tilde{x}(2s) = \cdots = \tilde{x}(k_1 s) = \tilde{x}(s_1 k)$. Thus, by (7) and Lemma 1, we have

$$\tilde{x}(t'+s) = \tilde{x}(pk_1 s + r) = \tilde{x}(ps_1 k + r) = \tilde{x}(r). \tag{9}$$

It follows from (8) and (9) that $\tilde{x}(t') = \tilde{x}(t'+s)$. Thus, $\tilde{x}(s+t) = \tilde{x}(t)$ for any $t \geq 0$. Then by Definition 2, the sufficiency is proved.

(iii) By Definition 2 and the result in (ii), the sufficiency is proved directly. Now, we prove the necessity. We only need to prove (6). For any $0 < T < s$, by (ii) of Definition 2, there exists $\tilde{t}$ such that $\tilde{x}(\tilde{t} + T) \neq \tilde{x}(\tilde{t})$. Then there exist integers $\bar{p}$ and $\bar{t}$ such that $\tilde{t} = \bar{p}s + \bar{t}$ $(0 \leq \bar{t} \leq s-1)$. From Definition 2, it follows that $\tilde{x}(\tilde{t} + T) = \tilde{x}(\bar{p}s + \bar{t} + T) = \tilde{x}(\bar{t} + T)$ and $\tilde{x}(\tilde{t}) = \tilde{x}(\bar{p}s + \bar{t}) = \tilde{x}(\bar{t})$. Thus, there exists $0 \leq \bar{t} \leq s-1$ such that $\tilde{x}(\bar{t} + T) \neq \tilde{x}(\bar{t})$. □

Remark 1. For TIBNs, if the cycle C shown in (4) has length s, it is required that $\tilde{x}(0), \tilde{x}(1), \ldots, \tilde{x}(s-1)$ are pairwise distinct (see Definition 5.4 in Cheng, Qi, Li, 2011). But for PTVBNs, this requirement is not necessary. For example, let $n = 1$, $f^1(x) = x$ and $f^2(x) = \neg x$ for (1). Then $0 \to 0 \to 1 \to 1$ is a cycle with length 4, while it is not pairwise distinct. This is an important property of PTVBNs different from that of TIBNs. If the considered BN is time-invariant, i.e. $k = 1$, then (5) exactly becomes the first condition $\tilde{x}(0) = \tilde{x}(s)$ in Definition 5.4 of Cheng, Qi, Li (2011). Moreover, (6) and $\tilde{x}(0) = \tilde{x}(s)$ imply that $\tilde{x}(0), \tilde{x}(1), \ldots, \tilde{x}(s-1)$ are pairwise distinct, which is just the other condition in the definition of cycles of TIBNs. Proposition 1 reveals some basic properties of cycles of PTVBNs.

3. Main results

In this section, we focus on revealing the relationship between the cycles of PTVBNs and the common cycles of a group of TIBNs.