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Scaled consensus[☆]

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ABSTRACT

A notion of scaled consensus is defined, wherein network components' states approach dictated ratios in the asymptote. A linear network dynamics that achieves scaled consensus for a prescribed interaction graph is introduced, and an equivalence with the class of (negative) singular M-matrices is explored. A few further characterizations of the scaled synchronization process are given, and finally a modification that allows tracking on the stable manifold is presented.

1. Introduction

An extensive literature has been developed on network consensus or synchronization processes, wherein network components' states reach a common initial-condition-dependent value via local interactions (Blondel, Hendrickx, Olshevsky, & Tsitsiklis, 2005; Ren, Beard, & Atkins, 2005; Wu & Chua, 1995). In many physical-world networks, components' states converge to an initial-condition-dependent equilibrium, but do not reach a common value: examples include compartmental mass-action systems, closed queueing networks, and water distribution systems (e.g., Haddad, Chellaboina, and Hui (2010) and Reiser (1981)). Likewise, many iterative network algorithms seek to assign diverse values across components—e.g., task-allocation and web-page-ranking algorithms (Page, Brin, Motwani, & Winograd, 1999).

Here, we interpret a class of network dynamics with initial-state-dependent equilibria as *scaled* consensus or synchronization processes. We first formalize a notion of scaled consensus wherein network components' scalar states reach assigned proportions, rather than a common value, in equilibrium. Then, a class of linear network processes is defined that achieve scaled consensus for any prescribed interaction topology. A one-to-one correspondence with (negative) singular M-matrices (Berman & Plemmons, 1979) is shown, which delineates the breadth of the defined class and aids temporal and spectral analysis. The processes' asymptotics are further analyzed, and a generalization to allow tracking on the stable manifold is described.

2. Problem formulation

A network with n components, labeled $1, \ldots, n$, is considered. A scalar state $x_i(t)$, $t \in R^+$, is modeled for each component $i \in 1, \ldots, n$. The state vector $\mathbf{x}(t) = \left[x_1(t), \ldots, x_n(t)\right]^T$ is assumed to be governed by a state-space differential equation $\dot{\mathbf{x}} = f(\mathbf{x})$, with the interactions between the components restricted by a directed graph (digraph) $\Gamma = (V = \{1, \ldots, n\}, E)$. Formally, for each component i, we define an *upstream neighbor set* $U(i)$ that lists all components j for which (j, i) is a directed edge in Γ. In the state equation, $\dot{x}_i$ is restricted to depend only on the local state $x_i(t)$ and the states $x_j(t)$ of the upstream neighbors ($j \in U(i)$).

The network process is said to achieve **scaled consensus**, if the ratios between the state variables reach specified constants in the asymptote. Formally, we say that the network achieves scaled consensus to $(\alpha_1, \ldots, \alpha_n)$, where the scalars $\alpha_1, \ldots, \alpha_n$ are assumed to be non-zero, if (1) the origin is not globally asymptotically stable but (2)

$$\lim_{t\to\infty} (\alpha_1 x_1(t) - \alpha_j x_j(t)) = 0 \tag{1}$$

for $j = 2, \ldots, n$, for all initial conditions $\mathbf{x}(0)$.

Remark 1. Scaled consensus can instead be defined in terms of the global asymptotic stability of the manifold specified by (1). The two notions are equivalent for linear processes.

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3. A linear scaled-consensus process

A family of linear network processes are defined, that achieve scaled consensus to $(\alpha_1, \ldots, \alpha_n)$ for a prescribed graph Γ. Specifically, let us consider the following state equations for the network components:

$$\dot{x}_i = sgn(\alpha_i) \sum_{j \in U(i)} k_{ij}(\alpha_j x_j(t) - \alpha_i x_i(t)) \tag{2}$$

for $i = 1, \ldots, n$, where $k_{ij} > 0$, and where sgn() returns +1 for positive arguments and −1 for negative arguments.

It is convenient to re-write (2) in vector form:

$$\dot{\mathbf{x}} = [diag(sgn(\alpha_i))]K[diag(\alpha_i)]\mathbf{x}, \tag{3}$$

where $[K]_{ij} = k_{ij}$ for $i = 1, \ldots, n, j \in U(i)$; $[K]_{ii} = -\sum_{j \in U(i)} k_{ij}$ for $i = 1, \ldots, n$; and $[K]_{ij} = 0$ otherwise. We also use the notation A for the state matrix of the process. Formal analysis of (3) requires some basic graph and matrix terminology. The graph Γ is said to be strongly connected, if there is a directed path from vertex i to vertex j, for any pair i and j. We say that a matrix C is commensurate with Γ if its off-diagonal entries satisfy: $[C]_{ij} \neq 0$ for $(j, i) \in E$ and $[C]_{ij} = 0$ for $(j, i) \notin E$. A matrix B that is commensurate with a strongly-connected graph is called irreducible.

The following lemma formalizes that the network process (2) achieves scaled consensus given connectivity:

Lemma 1. *For any strongly-connected graph Γ, the network process (2) achieves scaled consensus to $(\alpha_1, \ldots, \alpha_n)$.*

Proof. Let us consider the matrix $L = [diag(\alpha_i sgn(\alpha_i))]K$. This matrix L is irreducible, and has (1) non-negative off-diagonal entries and (2) zero row sums. Thus, L is the negative of a diffusive or (directed) Laplacian matrix on an irreducible graph. It follows immediately that L has a non-repeated eigenvalue $\lambda = 0$ with corresponding right eigenvector $\mathbf{v} = 1$, with its remaining eigenvalues strictly in the open left half of the complex plane (OLHP). However, noting that the state matrix A of the network process can be rewritten as $[diag(1/\alpha_i)][diag(\alpha_i sgn(\alpha_i))]K[diag(\alpha_i)]$, we see that L is similar to the state matrix A. It follows immediately that A has a non-repeated eigenvalue at $\lambda = 0$ with corresponding right eigenvector $\mathbf{v} = \left[1/\alpha_1, \ldots, 1/\alpha_n\right]^T$, and has all other eigenvalues in the OLHP. Thus, we have that

$$\lim_{t \to \infty} \mathbf{x}(t) = \left[1/\alpha_1, \ldots, 1/\alpha_n\right]^T \mathbf{w}^T \mathbf{x}(0), \tag{4}$$

where $\mathbf{w}^T$ is the left eigenvector of A associated with the zero eigenvalue (appropriately normalized), and $\mathbf{x}(0)$ is the initial state of the network process (3). The conditions for scaled consensus, that global asymptotic stability is not achieved but $\lim_{t\to\infty}(\alpha_1 x_1(t) - \alpha_j x_j(t)) = 0$, follow. □

Remark 2. Through selection of $\alpha_1, \ldots, \alpha_n$, the invariant manifold of the process (2) can be assigned to any one-dimensional subspace of R^n that is not perpendicular to a coordinate axis. Thus, the process can be designed to asymptotically reach the null space of a generic n-column matrix with $n - 1$ independent rows. With this observation, the process can be viewed as asymptotically enforcing generic linear constraints on the network components' states. Linear functionals of consensus-process states have also been considered in Sundaram and Hadjicostis (2008), however with a focus on computing the functionals rather than designing the stable manifold.

Remark 3. The scaled-consensus algorithm can be specialized to achieve multi-group consensus, wherein two or more partitions of the network achieve consensus to different values (Xia & Cao, 2011; Yu & Wang, 2010), through appropriate choice of $\alpha_1, \ldots, \alpha_n$.

4. Equivalence with negative singular *M*-matrices

A one-to-one correspondence between processes (2) with $(\alpha_1, \ldots, \alpha_n)$ in the positive orthant and (negative) singular M matrices is developed, which delineates the subset of linear scaled-consensus processes that are captured by (2), and permits model analysis using M-matrix constructs. To develop the correspondence, it is easy to verify that the state matrix A of the process (2) is a negative singular M matrix, for $(\alpha_1, \ldots, \alpha_n)$ in the positive orthant. Conversely, each (negative) singular M matrix commensurate with a strongly connected graph Γ specifies a dynamics of the form (2):

Lemma 2. *Consider any singular M-matrix R that is commensurate with the graph Γ. If Γ is strongly connected, the process $\dot{\mathbf{x}} = -R\mathbf{x}$ achieves scaled consensus, and further has the form (2) with $(\alpha_1, \ldots, \alpha_n)$ in the positive orthant.*

Proof. The matrix $-R$ can be written as $-sI + B$, where B is an irreducible nonnegative matrix commensurate with Γ, and s is a positive constant. From properties of irreducible nonnegative matrices, it follows that B has a non-repeated strictly-dominant eigenvalue μ with strictly-positive right eigenvector $\mathbf{v}$. Since R is singular and also an M-matrix, it follows that s equals $-\mu$. Thus, $-R$ is a matrix with (1) a non-repeated eigenvalue at 0 whose corresponding right eigenvector is $\mathbf{v}$, and (2) remaining eigenvalues in the OLHP.

Now consider the matrix $\widehat{K} = -R\,diag(\mathbf{v})$. This matrix is commensurate with Γ and also has non-negative off-diagonal entries. Additionally, we have that $\widehat{K}\mathbf{1} = -R\mathbf{v} = \mathbf{0}$, so $\widehat{K}$ has zero row sums. Thus, $-R = \widehat{K}(diag(\mathbf{v}))^{-1}$ is in the form (3), with $K = \widehat{K}$ and $\alpha_i = v_i^{-1} > 0$. Thus, we have shown that $\dot{\mathbf{x}} = -R\mathbf{x}$ defines a linear network process of the form (2), with $\alpha_i > 0$. From Lemma 1, it also follows that scaled consensus is achieved. □

The developed one-to-one correspondence has several implications. First, the result delineates the subset of linear scaled-consensus processes that are captured by our process model (2). Generically, a linear network process achieves scaled consensus if its state matrix (1) has eigenvalues in a closed-left-half-plane (CLHP) and (2) has a single eigenvalue on the imaginary axis (at 0) with non-zero eigenvector. From the fact that the state matrix has a null space, it readily follows that all linear scaled-consensus processes can be written in the form (2), if the requirement of positive edge-weights is relaxed. The additional requirement of positive edge weights encompasses all state matrices (corresponding to the specified graph) that are singular M-matrices and diagonal transformations thereof, for which the remaining eigenvalues are guaranteed to be in the open left-half-plane. While some processes with the positive-edge-weight condition relaxed may achieve scaled consensus, the set that does so is not easily defined since the correspondence to M-matrices cannot be relied on.

Second, the one-to-one correspondence allows application of M-matrix analyses in characterizing (2). For instance, the state matrices of processes (2) that achieve scaled consensus to a fixed $(\alpha_1, \ldots, \alpha_n)$ form a convex set, which may be useful in e.g. designing scaled-consensus processes. Temporal and spectral properties of the dynamics (2) are also implied. As one example, the orthant containing $(\alpha_1, \ldots, \alpha_n)$ are an invariant of the process dynamics (2) (Berman & Plemmons, 1979). Further implications are omitted in the interest of space.

For the special case of standard consensus (the case where the M-matrix R has a right eigenvector $\mathbf{v} = \mathbf{1}$), each diagonal entry in R equals the negative of the sum of the off-diagonal entries,