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On the nuclear norm heuristic for a Hankel matrix completion problem☆

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ABSTRACT

This note addresses the question if and why the nuclear norm heuristic can recover an impulse response generated by a stable single-real-pole system, if elements of the upper-triangle of the associated Hankel matrix are given. Since the setting is deterministic, theories based on stochastic assumptions for low-rank matrix recovery do not apply in the considered situation. A 'certificate' which guarantees the success of the matrix completion task is constructed by exploring the structural information of the hidden matrix. Experimental results and discussions regarding the nuclear norm heuristic applied to a more general setting are also given.

1. Introduction

Techniques of convex relaxation using the nuclear norm heuristic have become increasingly popular in the systems and control community, see e.g. the examples reported in Markovsky (2012), Vandenberghe (2012) and the discussions therein. This note provides a theoretical justification for the usage of the nuclear norm heuristic when it is applied to an fundamental task in systems theory, i.e. to recover the impulse response of a system from the first few entries of the related series. Precisely, we make the following assumptions throughout the note: (1) the provided entries are exact, i.e. there are no noise present, (2) the first n entries of the impulse response are provided while the last $n-1$ entries are to be completed.

The problem considered can be casted as a special case of the 'matrix completion' problem (Candés & Recht, 2009; Gross, 2011; Recht, Fazel, & Parrilo, 2010). However, in this work, the sampled entries are given deterministically, while the 'matrix completion' problems are typically analyzed using random sampling patterns. Furthermore, the underlying matrix is a structural (Hankel) matrix. These differences make the theories in the literature not applicable directly to this problem. While this task can be easily solved using standard techniques (Vandenberghe, 2012), the rationale for this work is that to provide a complete picture for understanding how the nuclear norm heuristic performs on this fundamental problem.

The following notational conventions will be used. Vectors are denoted in boldface, scalars are denoted in lowercase, matrices as capital letters, and sets are represented as calligraphic letters. $\mathcal{H}_n$ denotes the set of $n \times n$ Hankel matrices, I_n denotes the identity matrix of size $n \times n$, $\mathbf{e}_i$ denotes the unit vector with only the ith element to be one and all the other elements zero, $\|\cdot\|_*$ represents the nuclear norm (sum of all the singular values) of a matrix, $\|\cdot\|_2$ represents the spectral norm of a matrix, and $\|\cdot\|_F$ represents the Frobenius norm of a matrix.

2. Results

Theorem 1. *Given $-1 < h < 1$, define vector $\mathbf{h} \in \mathbb{R}^n$ as $\mathbf{h} = [1, h, h^2, \ldots, h^{n-1}]^T$, and matrix $G_0 \in \mathcal{H}_n$ as $\mathbf{h}\mathbf{h}^T$. Consider the following application of the nuclear norm heuristic:*

$$\hat{G}_0 \triangleq \arg\min_{G \in \mathcal{H}_n} \|G\|_* \quad \text{s.t. } G(i,j) = G_0(i,j), \quad \forall\, (i+j) \le n+1, \tag{1}$$

it holds that $\hat{G}_0$ is unique and $\hat{G}_0 = G_0$.

Remark 1. Since matrix G_0 is of rank one, when

$$\tilde{G}_0 \triangleq \arg\min_{G \in \mathcal{H}_n} \operatorname{rank}(G) \quad \text{s.t. } G(i,j) = G_0(i,j), \quad \forall (i+j) \le n+1, \tag{2}$$

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is solved, one has that $\tilde{G}_0 = G_0$, see e.g. Fazel, Hindi, and Boyd (2003) and Liu and Vandenberghe (2009).

Remark 2. When $|h|$ is close to 1, the condition number of G_0 is very large, which will cause numerical problems to the optimization problem.

For the case when $|h| > 1$, by running simulations, we find that the optimization will in general fail to recover G_0.

2.1. Proof of Theorem 1

Based on the matrices G_0 and G in Theorem 1, define:

$$H = G_0 - G. \tag{3}$$

Notice that by construction, all the entries of H in the upper triangle part are zero, so H can be decomposed as

$$H = \sum_{i=1}^{n-1} v_i G_i, \tag{4}$$

where $\{G_i\}_{i=1}^{n-1}$ are the basis matrices with the elements of the ith lower anti-diagonal equal to 1 and the others equal to zero and $v_i \in \mathbb{R}, \forall i = 1, \ldots, n-1$.

Define the projection matrix

$$P = \frac{G_0}{\|\mathbf{h}\|_2^2}$$

and its complement projection matrix as $Q = I_n - P$.

Proposition 1 will be used later, which characterizes the nuclear norm as the dual norm of the spectral norm for a given matrix (Recht et al., 2010).

Proposition 1. *Given $A \in \mathbb{R}^{n\times n}$ matrix, then*

$$\|A\|_* = \sup\{\mathrm{tr}(MA) : \|M\|_2 \le 1, M \in \mathbb{R}^{n\times n}\}. \tag{5}$$

The following result will be needed in Lemma 3.

Proposition 2. *Given H as defined in Eq. (3), if $H \neq 0$, then $QHQ \neq 0$.*

Proof. We prove that the only possibility for $QHQ = 0$ to hold is when $H = 0$. Notice that $H = (P+Q)H(P+Q)$, expanding this equality, we have that

$$H = PHP + PHQ + QHP + QHQ.$$

Hence if $QHQ = 0$, we have that

$$\begin{aligned} H &= PHP + PHQ + QHP \\ &= PH + QHP. \end{aligned}$$

Since $P = \frac{\mathbf{h}\mathbf{h}^T}{\|\mathbf{h}\|_2^2}$, the previous relation implies that H can be represented as $\mathbf{h}\mathbf{a}^T + \mathbf{b}\mathbf{h}^T$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Since H is symmetric, it holds that

$$\mathbf{h}\mathbf{a}^T + \mathbf{b}\mathbf{h}^T = \mathbf{a}\mathbf{h}^T + \mathbf{h}\mathbf{b}^T,$$

or equivalently

$$\mathbf{h}(\mathbf{b}-\mathbf{a})^T = (\mathbf{b}-\mathbf{a})\mathbf{h}^T. \tag{6}$$

Given the fact in Eq. (6), the two rank-one matrices $\mathbf{h}(\mathbf{b}-\mathbf{a})^T$ and $(\mathbf{b}-\mathbf{a})\mathbf{h}^T$ will have the same row space and column space, which implies that $\mathbf{b}-\mathbf{a} = k\mathbf{h}$, where $k = \frac{(\mathbf{b}-\mathbf{a})^T\mathbf{h}}{\|\mathbf{h}\|_2^2}$.

This implies that H can be written as

$$H = \mathbf{h}\mathbf{a}^T + \mathbf{b}\mathbf{h}^T = \mathbf{h}\mathbf{a}^T + \mathbf{a}\mathbf{h}^T + k\mathbf{h}\mathbf{h}^T,$$

i.e.

$$H = \left(\mathbf{a} + \frac{k}{2}\mathbf{h}\right)\mathbf{h}^T + \mathbf{h}\left(\mathbf{a} + \frac{k}{2}\mathbf{h}\right)^T.$$

Let $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T = \mathbf{a} + \frac{k}{2}\mathbf{h}$. Notice that the ith element of the first column of H equals $h^{i-1}c_1 + c_i$. By construction, the first column of H is a zero vector. Hence for $i = 1$, it holds that $2c_1 = 0$, i.e. $c_1 = 0$. Thus the ith element of the first column of H equals c_i, which implies that $c_2 = \cdots = c_n = 0$, i.e. $\mathbf{c} = 0$. This concludes that $H = 0$. □

Lemma 1 provides a sufficient condition for Theorem 1 to hold.

Lemma 1. *If for any $H \neq 0$ as in Eq. (3), one has that*

$$|\mathrm{tr}(PH)| < \|QHQ\|_*, \tag{7}$$

the optimization problem (1) recovers G_0 exactly.

Proof. Let $V \in \mathbb{R}^{n\times(n-1)}$ be a matrix which satisfies $VV^T = Q$ and $V^TV = I_{n-1}$. Noticing that any matrix with the row space orthogonal to the row space of matrix P and the column space orthogonal to the column space of matrix P can be represented as VBV^T, where $\|B\|_2 \le 1$, then the sub-gradients of $\|\cdot\|_*$ at G_0 are given as the set (see e.g. Recht et al., 2010):

$$\mathcal{S}_{\mathbf{h}} = \left\{P + VBV^T \ : \|B\|_2 \le 1\right\}. \tag{8}$$

By the property of sub-gradient, it holds that for any H as in Eq. (3),

$$\|G_0 + H\|_* \ge \|G_0\|_* + \langle H, F\rangle,$$

where $F \in \mathbb{R}^{n\times n}$ is any matrix which belongs to $\mathcal{S}_{\mathbf{h}}$.

Hence, for any H, if there exists a matrix in $\mathcal{S}_{\mathbf{h}}$, i.e. a B with $\|B\|_2 \le 1$, such that

$$\left\langle H, P + VBV^T\right\rangle > 0$$

or equivalently

$$\mathrm{tr}(HP) > \langle V^THV, -B\rangle, \tag{9}$$

then $\|G_0 + H\|_* > \|G_0\|_*$ holds, which implies Theorem 1.

We are left to find a matrix which satisfies inequality (9) given the assumption (7). From Eq. (7), we have that

$$|\mathrm{tr}(HP)| < \|QHQ^T\|_*,$$

and Q is a projection matrix onto an $n-1$ dimensional subspace, then

$$\|QHQ^T\|_* = \|V^THV\|_*,$$

which gives that

$$|\mathrm{tr}(HP)| < \|V^THV\|_*.$$

Furthermore, it follows from Proposition 1 that there exists a matrix B_1 with $\|B_1\|_2 \le 1$, such that

$$\|V^THV\|_* = \left\langle V^THV, B_1\right\rangle,$$

therefore it holds that

$$|\mathrm{tr}(HP)| < \left\langle V^THV, B_1\right\rangle.$$

Hence

$$\mathrm{tr}(HP) > -\left\langle V^THV, B_1\right\rangle = \langle V^THV, -B_1\rangle$$

holds, which gives that the inequality (9) holds for B_1. This concludes the proof. □

Next, we prove that the condition in Lemma 1 will always hold whenever $H \neq 0$. Lemma 2 constructs a matrix M_0 which will be used in Lemma 3 to constructs the 'certificate' M_1.