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Finite-Horizon Indefinite Mean-Field Stochastic Linear-Quadratic Optimal Control

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Abstract: For the finite-horizon indefinite mean-field stochastic linear-quadratic optimal control problems, the open-loop optimal control and the closed-loop optimal strategy are introduced and investigated together with their characterizations, difference and relationship. The open-loop optimal control can be defined for a fixed initial state, whose existence is characterized via the solvability of a linear mean-field forward-backward stochastic difference equations with stationary conditions. Differently, the closed-loop strategy is a global notion, which involves all the initial pairs. The existence of the closed-loop optimal strategy is shown to be equivalent to the solvability of a couple of generalized difference Riccati equations, the finiteness of the value function for all the initial pairs, and the existence of open-loop optimal strategy for all the initial pairs.

Keywords: Time-inconsistency, stochastic linear-quadratic optimal control, forward-backward stochastic difference equation

1. INTRODUCTION

In this paper, a kind of discrete-time stochastic linear-quadratic (LQ) optimal control problem of mean-field type is investigated. Compared with the classical stochastic optimal LQ control problem, an important feature of the problem is that both the objective functional and the dynamics involve the states and the controls as well as their expected values. In this case, the system dynamics is a discrete-time stochastic difference equation (SDE) of McKean-Vlasov type, which is also referred as the mean-field SDE (MF-SDE). As a feature of such a class of SDEs, the dynamics depend on the statistical distribution of the solution, which provides simple but effective techniques for studying large systems by reducing their dimension and complexity. This new feature roots itself in the category of the mean-field theory, which is developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamical systems. According to the mean-field theory, the interactions among agents are modeled by a mean-field term. When the number of individuals goes to infinity, the mean-field term approaches the expected value. The past few years have witnessed many successful applications of the mean-field formulation in various fields of engineering, games, finance and economics.

In this paper, the mean-field stochastic LQ optimal control with indefinite cost weighting matrices is studied. This problem is a natural generalization of those in Elliott *et al.* (2013), Ni *et al.* (2014). In fact, there is an increasing interest in the mean-field control theory in mathematics and control communities during the past years. The investigation of continuous-time mean-field stochastic differential equations could be traced back to the 1960s Mckean (1966). In Ahmed and Ding (2011), to cope with the possible time-inconsistency of optimal control, an extended version of dynamic programming principle is derived using the Nisio nonlinear operator semigroup. Subsequently, stochastic maximum principles are studied in several works Ahdersson and Djehiche (2011), Li (2012), which specify the necessary conditions for optimality. The results range from the case of a convex action space to the case of a general action space. As applications, the Markowitz mean-variance portfolio selection and a class of mean-field LQ problems are studied in Ahdersson and Djehiche (2011), Li (2012) using the stochastic maximum principle. In Yong (2013), the definite mean-field LQ control with

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a finite time horizon is systemically studied using a variational method and a decoupling technique. It is shown that the optimal control is of linear feedback form and that the gains are represented using solutions of two coupled differential Riccati equations. In Elliott *et al.* (2013), the discrete-time definite mean-field LQ problem is formulated as an operator stochastic LQ optimal control problem. By the kernel-range decomposition representation of the expectation operator and its pseudo-inverse, an optimal control is obtained based on the solutions of two Riccati difference equations. Wang *et al.* (2014) studies the stochastic maximum principle under partial information. Hafayed (2014) presents the maximum principle of mean-field type for the controlled mean-field forward-backward stochastic differential equations with Poisson jumps. For an extensive review of mean-field optimal control and other interesting aspects, we can also refer to the work Bensoussan *et al.* (2013). It is worth being mentioned that the study of controlled mean-field stochastic differential or difference equations is also partially motivated by a recent surge of interest in mean-field games Bardi and Priuli (2014), Bardi (2012), Bensoussan *et al.* (2013), Bensoussan *et al.* (2014), Carmona and Delarue (2013), Huang *et al.* (2003), Huang *et al.* (2007), Huang *et al.* (2012), Lasry and Lions (2007), Li and Zhang (2008), Tembine *et al.* (2014), Wang and Zhang (2012). Compared with the topic of this paper, mean-field games use decentralized controls, that is, the controls are selected to achieve each individual's own goal by using local information.

Indefinite stochastic LQ optimal control without mean-field terms was first studied at the end of last century. It is found that a stochastic LQ problem with indefinite cost weighting matrices may still be well-posed, which challenges the standard belief about LQ problems Ait Rmi *et al.* (2002), Ait Rami *et al.* (2001). It is further shown that indefinite stochastic LQ problems are closely related to Markowitz's mean-variance portfolio selection problems in financial investment Li and Ng (2000). As pointed out by Li and Ng (2000), when the expectation of state and control appear nonlinearly in cost functional, the corresponding problems are nonseparable in the sense that the standard dynamic-programming-based methodology fails to work. In Ni *et al.* (2014), for an inhomogeneous version of Problem (MF-LQ) with $L_k, \bar{L}_k = 0, k \in \mathbb{N}$, the authors propose a modified backward recursive technique, by which and the method of completing the square the authors get around the nonseparability. Moreover, it is shown that the well-posedness and the solvability of the mean-field LQ problem are equivalent to both the solvability of a set of GDREs and a constrained linear recursive equation. As an application, the multi-period mean-variance portfolio selection is well studied, and the obtained results extend those in Li and Ng (2000) to the case that the return rates of the risky securities are possibly degenerate.

In this paper, we shall study the finite-horizon indefinite mean-field LQ optimal control problems and give a more detailed and deeper investigation than that in Ni *et al.* (2014). Specifically, we introduce the open-loop optimal control and the closed-loop optimal strategy, and investigate their characterizations, difference and relationship. The open-loop optimal control can be defined for a fixed initial state (see Problem (MF-LQ) in Section II), whose existence is characterized via the solvability of a mean-field forward-backward stochastic difference equation (MF-FBSDE) and two stationary conditions. Therefore, the open-loop optimal control may depend on the initial state, which is then viewed as a local notion. Differently, the closed-loop optimal strategy is a global notion, which involves all the initial pairs $(l,h) \in L^2(\mathbb{N};\mathbb{R}^{n\times n}) \times L^2_{\mathcal{F}}(\mathbb{N};\mathbb{R}^m)$ (the notation is defined in Section II), and is independent of the initial pair. It is then shown that the existence of the closed-loop optimal strategy is equivalent to the solvability of a couple of GEREs, the finiteness of the value function for all the initial pairs, and the existence of open-loop optimal strategy for all the initial pairs.

2. INDEFINITE MEAN-FIELD LQ OPTIMAL CONTROL IN A FINITE HORIZON

2.1 Open-loop optimal control

Consider the following dynamic system

$$\begin{cases} x_{k+1} = (A_k x_k + \bar{A}_k \mathbb{E}x_k + B_k u_k + \bar{B}_k \mathbb{E}u_k) \\ \qquad\quad + (C_k x_k + \bar{C}_k \mathbb{E}x_k + D_k u_k + \bar{D}_k \mathbb{E}u_k) w_k, \\ x_0 = \zeta, \quad k \in \{0,1,...,N-1\} \triangleq \mathbb{N}, \end{cases} \tag{1}$$

where $A_k, \bar{A}_k, C_k, \bar{C}_k \in \mathbb{R}^{n\times n}$, and $B_k, \bar{B}_k, D, \bar{D}_k \in \mathbb{R}^{n\times m}$ are given deterministic matrices; $\mathbb{N}$ denotes the set $\{0,1,...,N-1\}$. In the sequel, we will denote $\{0,1,...,N\}$ by $\bar{\mathbb{N}}$. In (1), $\{x_k, k\in\bar{\mathbb{N}}\}$, $\{u_k, k\in\mathbb{N}\}$ and $\{w_k, k\in\mathbb{N}\}$ are the state, control and disturbance process, respectively; $\{w_k\}$ is assumed to be a martingale difference sequence defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and

$$\mathbb{E}[w_{k+1}|\mathcal{F}_k] = 0, \quad \mathbb{E}[(w_{k+1})^2|\mathcal{F}_k] = 1, \tag{2}$$

with $\mathcal{F}_k$ being the σ-algebra generated by $\{x_0, w_l, l = 0,1,\cdots,k\}$. For convenience, $\mathcal{F}_{-1}$ denotes $\sigma(x_0)$. The initial value ζ is assumed to be square integrable. The cost functional associated with (1) is

$$\begin{aligned} J(\zeta;u) = \sum_{k=0}^{N-1} \mathbb{E}\Big[& x_k^T Q_k x_k + (\mathbb{E}x_k)^T \bar{Q}_k \mathbb{E}x_k + 2x_k^T L_k u_k \\ & + 2(\mathbb{E}x_k)^T \bar{L}_k \mathbb{E}u_k + u_k^T R_k u_k + (\mathbb{E}u_k)^T \bar{R}_k \mathbb{E}u_k \Big] \\ & + \mathbb{E}\left(x_N^T G_N x_N\right) + (\mathbb{E}x_N)^T \bar{G}_N \mathbb{E}x_N, \end{aligned} \tag{3}$$

where $Q_k, \bar{Q}_k \in \mathbb{R}^{m\times m}, R_k, \bar{R}_k \in \mathbb{R}^{n\times n}, L_k, \bar{L}_k \in \mathbb{R}^{n\times m}, k\in\mathbb{N}, G_N, \bar{G}_N \in \mathbb{R}^{n\times n}$ are deterministic symmetric matrices. Let $L^2_{\mathcal{F}}(\{l,...,N-1\};\mathcal{H})$ be the set of $\mathcal{H}$-valued processes $z = \{z_k, k\in\{l,...,N-1\}\}$ such that z_k is $\mathcal{F}_{k-1}$-measurable and $\sum_{k=l}^{N-1}\mathbb{E}|z_k|^2 < \infty$. If $l = 0$, then $L^2_{\mathcal{F}}(\{l,...,N-1\};\mathcal{H})$ can be denoted by $L^2_{\mathcal{F}}(\mathbb{N};\mathcal{H})$. In addition, $L^2_{\mathcal{F}}(l;\mathcal{H})$ is the set of random variables ξ such that $\xi\in\mathcal{H}$ is $\mathcal{F}_{l-1}$-measurable and $\mathbb{E}|\xi|^2 < \infty$. Let $\mathcal{X}_0 = \{\zeta \,|\, \zeta \text{ is } \mathcal{F}_{-1}\text{-measurable and square integrable}\}$, which denotes the set of all the initial states. Then the optimal control with a finite time horizon is stated as follows.

Problem (MF-LQ). Given $\zeta\in\mathcal{X}_0$, find a $u^*\in\mathcal{U}_{ad}$ such that

$$J(\zeta;u^*) = \inf_{u\in L^2_{\mathcal{F}}(\mathbb{N};\mathbb{R}^m)} J(\zeta;u). \tag{4}$$