Electric Power Systems Research 158 (2018) 126–135

Contents lists available at ScienceDirect

Electric Power Systems Research

journal homepage: www.elsevier.com/locate/epsr

# Linear optimal power flow using cycle flows

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ARTICLE INFO

*Article history:*
Received 3 August 2017
Received in revised form 7 December 2017
Accepted 30 December 2017

*Keywords:*
Linear optimal power flow
DC power flow
Dual network
Graph theory

ABSTRACT

Linear optimal power flow (LOPF) algorithms use a linearization of the alternating current (AC) load flow equations to optimize generator dispatch in a network subject to the loading constraints of the network branches. Common algorithms use the voltage angles at the buses as optimization variables, but alternatives can be computationally advantageous. In this article we provide a review of existing methods and describe a new formulation that expresses the loading constraints directly in terms of the flows themselves, using a decomposition of the network graph into a spanning tree and closed cycles. We provide a comprehensive study of the computational performance of the various formulations, in settings that include computationally challenging applications such as multi-period LOPF with storage dispatch and generation capacity expansion. We show that the new formulation of the LOPF solves up to 7 times faster than the angle formulation using a commercial linear programming solver, while another existing cycle-base formulation solves up to 20 times faster, with an average speed-up of factor 3 for the standard networks considered here. If generation capacities are also optimized, the average speed-up rises to a factor of 12, reaching up to factor 213 in a particular instance. The speed-up is largest for networks with many buses and decentral generators throughout the network, which is highly relevant given the rise of distributed renewable generation and the computational challenge of operation and planning in such networks.



## 1. Introduction

Optimal power flow (OPF) problems can be constructed to find the welfare-maximizing generation and consumption levels in a network given the physical load flow equations, branch loading limits and generator cost functions. The full load flow equations are non-linear and the resulting optimization problem is non-convex, which makes it both challenging and computationally expensive to find a global optimum [1]. In transmission networks with sufficient reactive power compensation, linearizing the load flow equations introduces only small errors [2,3], with the benefit that the Linear OPF (LOPF) can be expressed as a linear problem, whose convexity guarantees that a local optimum is a global optimum.

LOPF algorithms are principally used in applications with high computational complexity where it would be impossible to use the full load flow equations, such as clearing markets with nodal pricing [4] (particularly with multi-period storage constraints and/or generator unit commitment), determining redispatch measures in markets with zonal pricing [5], optimizing dispatch taking account of contingencies (Security Constrained LOPF (SCLOPF)) [6,7] and in the long-term optimization of investment in generation and transmission assets [8,9]. Where higher accuracy solutions are required, linear solutions can be fed as an initial solution into algorithms that use the full non-linear load flow equations [1]. LOPF is becoming more important with the growth of renewable energy, since the fluctuating feed-in has led to more frequent situations where the network is highly loaded [10]. When large networks are optimized over multiple representative feed-in situations, especially with discrete constraints on generation dispatch, the LOPF problems can still take a significant time to solve, despite the linearization of the problem. Approaches in the literature to reducing the computational times of LOPF problems include decomposition [11–15], reformulating the problem using Power Transfer Distribution Factors (PTDFs) [16,17] and a parallelizable algorithm using the primal-dual interior point method [18].

In textbooks [6,19] and major software packages such as MATPOWER [20], DIgSILENT PowerFactory [21], PowerWorld [22] and PSAT [23], the linearization of the relations between power flows

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https://doi.org/10.1016/j.epsr.2017.12.034
0378-7796/

in the network and power injection at the buses is expressed indirectly through auxiliary variables that represent the voltage angles at the buses. In this paper, we introduce a new formulation of the LOPF problem that use the power flows directly, decomposed using graph theoretic techniques into flows on a spanning tree and flows around closed cycles in the network. The new formulation involves both fewer decision variables and fewer constraints than the angle-based formulation. We evaluate the computational performance of the various methods for the LOPF problem, showing that the cycle-based formulations can solve significantly faster than the traditional angle-based formulation. We examine not just the basic LOPF problem, but also applications that include more computationally challenging multi-period storage optimization and generation capacity expansion.

Cycle-flow techniques have already been used in [24] to improve the calculation times of PTDFs and to gain a new understanding of the propagation of line outages in networks [25]. The cycle-based LOPF formulation we call the 'Kirchhoff formulation' below was used in [26] for single-period LOPF and in [27] for single-period LOPF with optimal transmission switching; in contrast to those papers, here we provide an additional new cycle-based formulation and benchmark both formulations against established formulations for a different set of computationally-challenging problems: those extending over multiple periods.

In Section 2 the different formulations of the linear load flow are reviewed to prepare for the introduction of the optimization in Section 3. Extensions beyond the basic LOPF problem are described in Section 4 and the results of the performance analysis are presented in Section 5. Variables are defined in Table 1.

## 2. Linear load flow formulations

The aim of the linear load flow calculation is to calculate the active power flow $f_\ell$ on each of the branches $\ell=1, \ldots, L$ in terms of the active power $p_i$ injected or consumed at each of the buses $i = 1, \ldots, N$. In this section four methods are presented for solving the linear load flow, which lead to different formulations of the LOPF problem, as discussed in the next section. The different formulations lead to mathematically identical solutions, as demonstrated in this section.

The linear approximation is valid if all branch resistances $r_\ell$ are negligible compared to the branch reactances $x_\ell$, $r_\ell \ll |x_\ell|$, reactive power flows may be neglected, all voltage magnitudes are kept at nominal value and if all voltage angle differences across branches $\theta_\ell$ are small enough that we can approximate $\sin\theta_\ell \sim \theta_\ell$. The usefulness of the linear approximation and the errors thereby introduced are discussed in [2,3]. If the approximation holds, the real power over a transmission line $\ell$ is given by

$$f_\ell = \frac{\theta_\ell}{x_\ell}, \quad (1)$$

where $\theta_\ell$ is the voltage angle difference between the terminal buses of line $\ell$.

The flows $f_\ell$ are constrained to be physical by the two Kirchhoff circuit laws for the current and voltage. Kirchhoff's Current Law (KCL) states that the current injected at each bus must equal the current withdrawn by the branches attached to the bus. This law can be expressed using the incidence matrix $K_{i\ell}$, which has non-zero values +1 if branch $\ell$ starts on bus $i$ and −1 if branch $\ell$ ends on bus $i$. KCL then reads

$$p_i = \sum_\ell K_{i\ell} f_\ell \quad \forall i = 1, \ldots, N. \quad (2)$$

KCL directly implies power conservation $\sum_i p_i = 0$ because $\sum_i K_{i\ell} = 0$ for all lines $\ell$. KCL provides $N$ linear equations for the $L$ unknown flows $f_\ell$, of which one is linearly dependent. This is not sufficient to uniquely determine the flows unless the network is a tree. Hence, $L-N+1$ additional independent equations are needed.

The necessary equations and physicality are provided by the Kirchhoff Voltage Law (KVL), which states that the sum of potential differences across branches around all cycles in the network must sum to zero. It follows from graph theory that there are $L-N+1$ independent cycles for a connected graph [28], which provides enough equations to constrain the $f_\ell$ completely. The independent cycles $c \in \{1, \ldots L-N+1\}$ are expressed as a directed linear combination of the branches $\ell$ in the cycle incidence matrix

$$C_{\ell c} = \begin{cases} 1 & \text{if edge } \ell \text{ is element of cycle } c, \\ -1 & \text{if reversed edge } \ell \text{ is element of cycle } c, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Then the KVL becomes

$$\sum_\ell C_{\ell c}\theta_\ell = 0 \quad \forall c = 1, \ldots, L-N+1. \quad (4)$$

where $\theta_\ell = \theta_i - \theta_j$ is the angle difference between the two buses $i$, $j$ which branch $\ell$ connects. Using Eq. (1), KVL can be expressed in terms of the power flows as

$$\sum_\ell C_{\ell c} x_\ell f_\ell = 0 \quad \forall c = 1, \ldots, L-N+1. \quad (5)$$

### 2.1. Angle formulation

Commonly, the linear load flow problem is formulated in terms of the voltage phase angles $\theta_i$, $i \in \{1, \ldots, N\}$. Using the incidence matrix the power flows are expressed as

$$f_\ell = \frac{1}{x_\ell}\sum_i K_{i\ell}\theta_i \quad \forall \ell = 1, \ldots, L \quad (6)$$

If the $L \times L$ diagonal matrix $B$ is defined with $B_{\ell\ell} = 1/x_\ell$ then the KCL equation (2) becomes

$$\begin{aligned} p_i &= \sum_{\ell,k,j} K_{i\ell} B_{\ell k} K_{jk} \theta_j \\ &= \sum_j \Lambda_{ij}\theta_j, \quad \forall i = 1, \ldots, N, \end{aligned} \quad (7)$$

**Table 1**
Variable definitions.

| Variable | Definition |
|---|---|
| $i, j \in \{1, \ldots N\}$ | Bus labels |
| $s \in \{1, \ldots G\}$ | Generation source labels (wind, solar, gas, etc.) |
| $k, \ell \in \{1, \ldots L\}$ | Branch labels |
| $c, d \in \{1, \ldots L-N+1\}$ | Cycle labels |
| $t \in \{1, \ldots T\}$ | Snapshot/time point labels |
| $d_{i,s}$ | Dispatch of generator at bus $i$ with source $s$ |
| $D_{i,s}$ | Available power of generator $i$, $s$ |
| $l_i$ | Electrical load at bus $i$ |
| $\theta_i$ | Voltage angle at bus $i$ |
| $p_i$ | Total active power injection |
| $\theta_\ell$ | Voltage angle across a branch |
| $f_\ell$ | Branch active power flow |
| $g_\ell$ | Flow on spanning tree (zero if $\ell$ not in tree) |
| $h_c$ | Flow around cycle $c$ |
| $F_\ell$ | Branch active power rating |
| $x_\ell$ | Branch series reactance |
| $K_{i\ell}$ | $N \times L$ incidence matrix |
| $C_{\ell c}$ | $L \times (L-N+1)$ cycle matrix |
| $T_{\ell i}$ | $L \times N$ tree matrix |
| $B_{\ell k}$ | Diagonal $L \times L$ matrix of branch susceptances |
| $\Lambda$ | $N \times N$ weighted Laplacian matrix $\Lambda = KBK^T$ |