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In-Block Controllability of Controlled Switched Linear Systems on Polytopes ⋆

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Abstract: The paper introduces the study of in-block controllability (IBC) of controlled switched linear systems on polytopes, which formalizes controllability of controlled switched linear systems under state constraints. In particular, for a given switched linear system and a given polytope, representing the state constraints, we study whether all the states in the interior of the polytope are mutually accessible through its interior using proper switching functions and uniformly bounded control inputs. By studying the geometry of the problem, we provide three necessary conditions for IBC. Then, we provide two cases where these necessary conditions are also sufficient. We also give illustrative examples of the main results.

1. INTRODUCTION

For a given controlled switched linear system and a given full-dimensional polytope X, representing the state constraints, we study whether all the states in the interior of X are mutually accessible through its interior by using proper switching functions satisfying some standard conditions Yang (2002), and uniformly bounded control inputs.

We summarize the motivations behind the proposed study as follows. First, controlled switched linear systems have received special interest Sename et al. (2013); Yang (2002); Stikkel et al. (2004); Sun et al. (2002); Morse (1993) because of their wide range of applications. Simple examples may include linear electric circuits with switches, mechanical systems with relays, and hydraulic systems with on/off valves, among others. Our study extends the controllability results of Yang (2002); Stikkel et al. (2004); Sun et al. (2002) to switched linear systems under state constraints, which is important in many practical scenarios when one has to take the safety and performance constraints of the system into consideration. Second, in Helwa, Caines (2014a), we studied conditions for IBC of affine systems on polytopes, and then used this study in Helwa, Caines (2014b) to study controllability and build hierarchical control structures of piecewise affine (PWA) hybrid systems, with only autonomous switchings that may happen when the state trajectories of the affine systems cross prescribed facets of the polytopes. In this paper we extend the results of Helwa, Caines (2014a) to the case of controlled switched linear systems, which is the first step in generalizing the results of Helwa, Caines (2014b) to the case when we have both autonomous and controlled switchings.

The IBC notion was first introduced in Caines, Wei (1995) for finite state machines, and was then extended to nonlinear systems on closed sets and to automata in Caines, Wei (1998) and Hubbard, Caines (2002), respectively. In these papers, the IBC notion was used to build hierarchical structures of the systems, but constructive conditions for the IBC property to hold were not studied. In Helwa, Caines (2014a), we provided easily checkable necessary and sufficient conditions for IBC of affine systems on polytopes, and then used these conditions to study mutual accessibility properties of PWA hybrid systems Helwa, Caines (2014b) and nonlinear systems Helwa, Caines (2015). In Helwa, Caines (2014c), we extended the IBC notion to the case where we have soft and hard safety constraints.

In this paper, we extend the in-block controllability study in Helwa, Caines (2014a) to the important class of controlled switched linear systems. After defining IBC of switched linear systems on polytopes in Section 2, we provide three necessary conditions for IBC in Section 3. We then explore in Section 4 two cases where these necessary conditions are also sufficient. For the mathematical background, refer to Section II of Helwa, Caines (2014a). In our study, we make use of the geometric tools developed for the control-to-facet problem (also called the reach control problem) on polytopes Habets, van Schuppen (2004); Broucke (2010); Helwa, Broucke (2012, 2013, 2014, 2015).

Notation. Let $K \subset \mathbb{R}^n$ be a set. The closure denotes $\overline{K}$, the interior is K°, and the boundary is ∂K. The notation $K_1 \setminus K_2$ denotes the elements of the set K_1 not in K_2. For vectors $x,\ y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of the two vectors. The notation $\|x\|$ denotes the Euclidean norm of x. The notation $\mathbb{R}^+$ denotes the set of nonnegative real numbers. The notation co $\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. Finally, B_0 denotes the unit open ball centered at the origin, and $B_\delta(x)$ denotes the open ball of radius δ centered at x.

2. IN-BLOCK CONTROLLABILITY

In this section we define the in-block controllability of controlled switched linear systems on polytopes. Consider an n-dimensional polytope $X := \text{co}\,\{v_1, \ldots, v_p\}$ with vertex set $\{v_1, \ldots, v_p\}$ and facets $F_1, \ldots, F_r$. Let h_i be the unit normal to the facet F_i pointing outside the polytope.

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Let $L < \infty$ and $N := \{1, \cdots, L\}$. Consider the following controlled switched linear system:

$$\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u\,, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $\sigma : \mathbb{R}^+ \to N$ is a piecewise constant switching function. The matrices $A_{\sigma(t)}$ and $B_{\sigma(t)}$ are piecewise constant matrices, whose values depend on σ, i.e. $A_{\sigma(t)} : N \to \mathbb{R}^{n\times n}$ and $B_{\sigma(t)} : N \to \mathbb{R}^{n\times m}$. If $\sigma(t)$ switches at time instant t_i from a value $n_{i-1} \in N$ to a value $n_i \in N$, where $n_{i-1} \neq n_i$, for $i = 1, 2, \cdots, k$, we call $\{(n_{i-1}, t_i, n_i) : i = 1, \cdots, k\}$ the time-mode switching set, where k can be finite or infinite. Consequently, we call the ordered set $\{t_i : i = 1, \cdots, k\}$ the switching time set and $\{n_i : i = 0, \cdots, k\}$ the switching mode set. We have the following standard assumption Yang (2002); Stikkel et al. (2004).

Assumption 2.1. (i) $\sigma(t)$ is left-continuous, and any time interval within which $\sigma(t)$ is constant is no less than a proper positive scalar $T_d > 0$, called the dwell time. (ii) The switching time set $\{t_i : i = 1, \cdots, k\}$, the corresponding switching mode set $\{n_i : i = 0, \cdots, k\}$, and the control input $u(t)$ within each selected mode are determined by the control design. (iii) There are no discontinuous state jumps during mode switches.

Assumption 2.1(i) ensures that there is no Zeno/Chattering Phenomenon Yang (2002). We say a switching function is *proper* if it satisfies Assumption 2.1. Throughout the paper, we assume that the control input $u(t) : [0, \infty) \to \mathbb{R}^m$ is measurable and bounded on any compact time interval, which ensures the existence and uniqueness of the solutions of the linear system within each selected mode. Let $\phi(x_0, t, u, \sigma)$ denote the trajectory of (1) under a control law u and a switching function σ, with initial condition x_0, and evaluated at time instant t.

We say that (1) is *(completely) controllable* if for every $x_0,\ x_f \in \mathbb{R}^n$, there exist a control input $u(t)$ and a proper switching function $\sigma(t)$ that steer the system (1) from x_0 to x_f in finite time. For studying controllability of controlled switched linear systems, it was shown that the Lie algebra techniques, used in the framework of nonlinear systems, can be of special use. With the aid of these techniques, it was shown that although controlled switched linear systems are time-varying systems, their controllability can be determined from the matrices of the linear systems Sename et al. (2013), Sun et al. (2002).

Theorem 2.1. (Proposition 11, Ch 3, Sename et al. (2013)).

The controlled switched linear system (1) is controllable if and only if $\mathcal{R}_{A,B} = \mathbb{R}^n$, where

$$\mathcal{R}_{A,B} := \text{span}\{(\prod_{j=1}^{J} A_{l_j}^{i_j})B_k \ : \ k \in \{1, \cdots, L\}\,,$$
$$J \geq 0,\ l_j \in \{1, \cdots, L\}\,,\ i_j \in \{0, \cdots, n-1\}\}.$$

Moreover, if one considers the finitely generated Lie algebra containing $A_1, \cdots, A_L$ and a basis $\hat{A}_1, \cdots, \hat{A}_{L'}$ of this algebra, then

$$\mathcal{R}_{A,B} = \sum_{k=1}^{L}\sum_{n_1=0}^{n-1}\cdots\sum_{n_{L'}=0}^{n-1} \text{Im}\ (\hat{A}_1^{n_1}\cdots\hat{A}_{L'}^{n_{L'}}B_k).$$

Notice that $\mathcal{R}_{A,B}$ is the smallest invariant subspace for all the A_i's containing $\bigcup_{i=1}^{L} \text{Im}\ (B_i)$, the image of B_i.

In this paper, we are interested in studying controllability of controlled switched linear systems under state constraints. In particular, for a given n-dimensional polytope X satisfying $0 \in X^\circ$, we study under (1) mutual accessibility of the states in X° through X°, which can be formally stated as follows (after Caines, Wei (1998)).

Definition 2.1. (In-Block Controllability (IBC)). Consider the controlled switched linear system (1) on an n-dimensional polytope X satisfying $0 \in X^\circ$. We say that (1) is *in-block controllable (IBC)* with respect to (w.r.t.) X if there exists $M > 0$ such that for all $x,\ y \in X^\circ$, there exist $T \geq 0$, a proper switching function $\sigma(t)$ defined on $[0, T]$, and a control input $u(t)$ defined on $[0, T]$ such that (i) $\|u(t)\| \leq M$ and $\phi(x, t, u, \sigma) \in X^\circ$ for all $t \in [0, T]$, and (ii) $\phi(x, T, u, \sigma) = y$.

Notice that in Definition 2.1, we assume that initially one can select any switching mode n_0, and that there is no constraint on the selected final switching mode n_f. This is different from Yang (2002) which studies the reachability of the hybrid state (n, x), where $n \in N$ and $x \in \mathbb{R}^n$, from (n_0, x_0) to (n_f, x_f). Also, notice that if we have the switching time set $\{t_i : i = 1, \cdots, k\}$ such that $0 =: t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} := T$, then by Assumption 2.1 (i), we have $t_{i+1} - t_i \geq T_d$, the dwell time, for $i = 0, \cdots, k$.

3. NECESSARY CONDITIONS

We investigate necessary conditions for IBC of (1) w.r.t. a polytope X satisfying $0 \in X^\circ$. First, controllability of (1) is a necessary condition for IBC.

Theorem 3.1. Consider the system (1) defined on an n-dimensional polytope X satisfying $0 \in X^\circ$. If the system (1) is IBC w.r.t. X, then (1) is controllable.

Proof. Omitted for brevity.

Next, in Helwa, Caines (2014a), other two necessary conditions for IBC of affine systems on polytopes were identified, namely the invariance conditions and the backward invariance conditions. Here, we investigate whether we have similar necessary conditions for the case of controlled switched linear systems. To that end, let $\lambda > 0$, and define $\lambda X := \{x \in \mathbb{R}^n : x = \lambda y,\ y \in X\}$, i.e. λX is a λ-scaled polytope of the polytope X satisfying $0 \in X^\circ$.

Lemma 3.2. Consider the controlled switched linear system (1) and an n-dimensional polytope X satisfying $0 \in X^\circ$. If (1) is IBC w.r.t. X, then (1) is IBC w.r.t. λX for any $\lambda > 0$.

For a linear subsystem of (1), let $\mathcal{B}_i := \text{Im}\ (B_i)$, the image of B_i, and define the set of possible equilibria of the linear system $\mathcal{O}_i := \{\ x \in \mathbb{R}^n \ : \ A_i x \in \mathcal{B}_i\ \}$. At any $x \in \mathcal{O}_i$, the vector field of the linear system $\dot{x} = A_i x + B_i u$ can vanish by a proper selection of the input u. Also, if $\bar{x}_i$ is an equilibrium point of the linear system under any control $\bar{u}_i$, then $\bar{x}_i \in \mathcal{O}_i$ Broucke (2010). Next, let $J := \{1, \cdots, r\}$, and $J(x) := \{j \in J : x \in F_j\}$. That is, J is the set of indices of the facets of X, and $J(x)$ is the set of indices of the facets of X in which x is a point. We define the closed, convex tangent cone to the polytope X