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A Study on Solving Guard and Invariant Set Intersection in Zonotope-based Reachability of Linear Hybrid Systems

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Abstract: Solving the problem of guard and invariant intersection when deciding for zonotopes to represent reachable sets of linear hybrid systems is still challenging. This is due to the fact that the intersection operation causes the loss of the zonotopic form. Furthermore a tradeoff must be found between computational efficiency and the tightness of the approximation. In this paper, we provide an overview, improve on some methods as well as carry out a comparative evaluation of different methods for zonotope/hyperplane intersection.

These methods are then evaluated in combination with different clustering techniques in the context of guard intersection for the reachability analysis of two-tank and colliding-masses benchmarks. We thereafter propose zonotope/halfspace and zonotope/polyhedron intersection methods as solution for handling invariants inside continuous modes. Experimental evaluation however reveals that embedded in the reachability computational process, these methods do differ in performances as compared to when they are used as standalone functions.

Keywords: Reachability, Hybrid Systems, Zonotope, Polyhedron, Dichotomous search, Singular Value Decomposition,, Intersection Bundle, Clustering.

1. INTRODUCTION

Over the last few years, there has been an increasing interest in the reachability analysis of hybrid systems due to its intrinsic usefulness for the design and verification of hybrid systems. Owing to the fact that the computation of the reachable sets of most systems is undecidable, approximations are generally proposed to effectively deal with the problem. These reachable sets can be represented in various ways, such as by support functions, polytopes, ellipsoids, hyper-rectangles and zonotopes.

However, against the backdrop of developments in modelling complex hybrid systems for industrial usage, one of assessment criteria for the applicability of an approximated reachable set should be its scalability to accommodate systems of higher dimensionality. Methods using representations such as polytopes (Hwang et al. (2005); Min et al. (2009)), ellipsoids (Kurzhanski and Varaiya (2000)) and hyperrectangles (Stursberg and Krogh (2003)) have been proven to be computationally expensive and remain confined largely to the use in small systems. Zonotopes, on the other hand, have been found to be advantageous in control theory in representing orbits without a 'wrapping effect' — an exponential growth in the over-approximations of orbit enclosures across iterations. Furthermore, unlike support functions, reachability computation using zonotopes does not require the solving of an optimization problem.

Apart from the complexity of reachable set computation inside continuous modes, the handling of intersections of reachable sets with guard transitions is also a challenge. While using zonotopes to represent reachable sets has been proven to be efficient (Girard and Guernic (2008)), an intersection with guard leads to a loss of the zonotopic form on one hand and requires particular adequate techniques to find the intersection on the other.

The problem of handling transitions with zonotopes has been tackled in Girard and Guernic (2008) for hyperplane guards where the multidimensional intersection problem is transformed into a series of two-dimensional problems by applying projections. A solution based on zonotope/polytope transformation has been suggested in Althoff et al. (2010) to handle polytopic guards by computing a tight parallelotopic over-approximation of the zonotope. Subsequently the intersection is computed using algorithms in the MPT-toolbox. The resulting polytope is lastly over-approximated using parallelotopes.

In the field of control theory, research has been conducted in determining problem of output feedback control using a zonotopic set-membership estimation. This is often based on the computation of an outer approximation of the intersection of the zonotopic consistent state set with the predicted state. This guarantees the enclosure of all the elements of the output vector simultaneously. The predicted set, however, can take the form of a strip or generally given as a polytope. Algorithms for zonotope/strip intersection based on minimizing the segments or the volume of the resulting zonotope have been presented in Alamo et al. (2005). An extension of the segment based approach was thereafter improved to handle zonotope/polytope inter-

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section (Tabatabaeipour and Stoustrup (2013); Le et al. (2013)). In addition, further guaranteed state intersection methods for zonotope/hyperplane and zonotope/zonotope intersections using Singular Value Decomposition (SVD) have been proposed in Lalami (2008).

In this work, we consider these methods of intersection computation and compare them in the context of a guard transition in hybrid systems while improving on certain intersection methods in section 3. Subsequently, we show that it is possible to include invariants into the continuous dynamics of the hybrid system with an adequate adaption of the state estimation algorithms for zonotope/strip and zonotope/polyhedron guaranteed intersection in section 7. We also investigate various strategies for handling the intersection bundles of zonotopes intersecting the guard in section 6 and discuss their advantages and drawbacks. Lastly in section 8, these methods are then evaluated as standalone functions with randomly generated 3D-zonotopes and hyperplanes. Intersection methods are also incorporated in the reachability computation process and benchmarked. We draw conclusions from the results of these benchmarks on the feasibility and efficiency of the different combinations of various methods.

2. ZONOTOPES — DEFINITIONS AND PROPERTIES

A zonotope Z of order $r = p/n$ is defined by its center $c \in \mathbb{R}^n$ and its generators $g_1, \dots, g_p \in \mathbb{R}^n$

$$\begin{aligned} Z &= (c, \langle g_1, \dots, g_p \rangle) \\ &= \{x \in \mathbb{R}^n \mid x = c + \sum_{i=1}^{p} s_i g_i,\ -1 \le s_i \le 1\} \end{aligned} \tag{1}$$

Zonotopes possess several properties that make them particularly attractive for their use as approximating set for reachability analysis.

2.1 Linear image

Let $\mathcal{L}$ be a linear transformation and $Z = (c, \langle g_1, \dots, g_p \rangle)$ a zonotope. The linear image of a zonotope is given as follows

$$\mathcal{L}Z = (\mathcal{L}c, < \mathcal{L}g_1, \dots, \mathcal{L}g_p >). \tag{2}$$

2.2 Minkowski sum and convex hull

Let $Z_1 = (c_1, \langle g_1, \dots, g_p \rangle)$ and $Z_2 = (c_2, \langle h_1, \dots, h_q \rangle)$ be two zonotopes. The Minkowski sum of two zonotopes is given by

$$Z_1 \oplus Z_2 = (c_1 + c_2, \langle g_1, \dots, g_p, h_1, \dots, h_q \rangle). \tag{3}$$

The convex hull $CH(Z_1 \cup Z_2)$ can be over-approximated with the following zonotope Z_{CH}

$$\frac{1}{2}\begin{cases} (c_1 + c_2, \langle g_1 + h_1, \dots, g_p + h_p, c_1 - c_2, \\ \qquad g_1 - h_1, \dots, g_p - h_p \rangle) & \text{if } p = q \\ (c_1 + c_2, \langle g_1 + h_1, \dots, g_q + h_q, c_1 - c_2, \\ \quad g_1 - h_1, \dots, g_q - h_q, 2g_{q+1}, \dots, 2g_p \rangle) & \text{if } p > q \\ (c_1 + c_2, \langle g_1 + h_1, \dots, g_p + h_p, c_2 - c_1, \\ \quad h_1 - g_1, \dots, h_p - g_p, 2h_{p+1}, \dots, 2h_q \rangle) & \text{if } p < q \end{cases} \tag{4}$$

These operations involve vector concatenation, leading consequently to a significant increase in the number of generators of the over-approximating zonotope. This thus calls for the reduction of the order of zonotope to control complexity.

2.3 Order reduction

To control the complexity of computation, a maximum zonotope order is fixed in advance. A zonotope $Z = (c, \langle g_1, \dots, g_{rn} \rangle)$ of order r can be reduced to a zonotope of order q by replacing the $n(r - q + 1)$ less significant generators with their interval hull. Now for a zonotope $Z = \big(c, \langle g_1, \dots, g_{(q+1)n} \rangle\big)$ satisfying

$$\|g_1\|_1 - \|g_1\|_\infty \le \dots \le \big\|g_{(q+1)n}\big\|_1 - \big\|g_{(q+1)n}\big\|_\infty . \tag{5}$$

the order is reduced by one by constructing a new zonotope $\widehat{G} = \big(c, \langle Q, g_{2n+1}, \dots, g_{(q+1)n} \rangle\big)$ where $Q \in \mathbb{R}^{n \times n}$ is a diagonal matrix with

$$Q_{ii} = \sum_{j=1}^{2n} \left|g_j^i\right|,\ i = 1, \dots, n$$

g_j^i is the $i^t h$ component of g_j. The obtained zonotope has hence order q and satisfies $Z \subseteq \widehat{Z}$ (Girard (2005)).

3. INTERSECTION OF A ZONOTOPE AND A HYPERPLANE GUARD TRANSITION

In the field of hybrid systems, a guard transition can be represented as a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : \langle d, x \rangle = e\}$, $d \in \mathbb{R}^n$ and $e \in \mathbb{R}$. At every iteration of the reachability analysis algorithm, we are required to check if the current reachable set intersects with a guard. This is a computationally easy problem defined as such, for a zonotope $Z = \langle c; g_1, ..., g_p \rangle$

$$Z \cap \mathcal{H} \Leftrightarrow \langle c, d \rangle - \sum_{i=1}^{p} |\langle g_i, d \rangle| \le e \le \langle c, d \rangle + \sum_{i=1}^{p} |\langle g_i, d \rangle| . \tag{6}$$

For the computation of the approximated intersection, we introduce two existing methods, one used specifically in context of reachability analysis and the other originally in control theory.

3.1 ND-2D Projection

In Girard and Guernic (2008), the problem of computing an over-approximation of the intersection of a zonotope Z with a hyperplane $\mathcal{H}$ is reduced to the problem of computing of several intersections of a 2-dimensional zonotope (zonogon) with a plane. For this purpose, a base $D = \{l_1, \dots, l_q\}$ of $\mathcal{H}$ is computed. For each $j \in \{1, \dots, q\}$, we compute the zonogon $Z_j^{(2)} = \mathbb{M}_{d,l_j} Z$ where $\Pi_{(d,l_j)}$ is the projection on the plane defined by the vector pair (d, l_j) defined as follows

$$\begin{aligned} \Pi_{(d,l_j)} &: \mathbb{R}^n \to \mathbb{R}^2 \\ &\quad x \longmapsto (\langle d, x \rangle ; \langle l_j, x \rangle) = \mathbb{M}_{d,l_j} x \end{aligned} \tag{7}$$

The projection of the hyperplane $\mathcal{H}$ for each j is the vertical line $L = \big\{u = (x; y) \in \mathbb{R}^2 :\ x = e\big\}$. The next step consists of computing a supremum point $(e; M_j)$ and an infinimum point $(e; m_j)$, corresponding to the intersection of the zonogon Z_j^2 with the straight line L in two-dimension