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Bifurcations in Timed Continuous Petri Nets

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Abstract: This paper is concerned with the steady-state throughput changes due to firing rate variations in Timed Continuous Petri Nets (TCPN) under infinite server semantics. Jumps in this value (equivalently in the steady state marking) are bifurcations. It presents a novel structural characterization of bifurcations based on the P-semiflows and T-semiflows appearing in the TCPN configurations. In order to give this characterization, first strongly connected and conservative JF-TCPN (join free TCPN) are studied. Bifurcations are not possible in this class of nets. Afterwards, strongly connected and consistent CF-TCPN (choice free TCPN) are addressed. It is shown that this class of nets does not exhibit bifurcations. This kind of result is extended to topologically Equal-conflict nets when we vary the firing rates of transitions in conflict in the same proportion. Based on these results, bifurcations in MTS-TCPN (Mono-T-Semiflow TCPN) including non-equal conflicts and joins are addressed. The main result in this class of nets states that bifurcations appear when the places restricting the transition's flow in a certain configuration do not belong to the support of any net P-semiflow. Finally, this work presents a relationship between bifurcations and non monotonic throughput in this class of systems.

1. INTRODUCTION

Petri nets (PNs) are a well-known formalism to model and analyze discrete event systems. However, when PNs are heavily marked, the state explosion problem appears limiting the net analysis capability. In order to overcome it, different approaches have been proposed. *Continuization* is an approximation technique that relaxes the integrality of marking and the transitions are fired in positive real amounts leading to *continuous Petri nets* (CPN) [Alla and David, 1998], [Silva and Recalde, 2002]. These nets were enriched with the addition of time, allowing the study of performance and reliability of complex systems. Time was associated to transitions as firing rates resulting in *timed continuous Petri nets* (TCPNs). In this work the *infinite server semantics* is used [Silva and Recalde, 2002]. Under this consideration TCPNs evolve as positive switched linear systems.

TCPNs are useful to model and analyze biological systems [Ross-León et al., 2010], traffic intersections [Julvez and Boel, 2010], productions systems [Silva et al., 2014]. A lot of research has been done in characterizing observability [Mahulea et al., 2010], [Aguayo-Lara et al., 2014], and controllability [Vázquez et al., 2014], [Mahulea et al., 2008] in these nets.

The study of bifurcations in TCPN, however, is very recent. [Júlvez et al., 2005] showed that the naive idea of increasing the firing rate of transitions do not necessarily increase the net system throughput. Surprisingly it can present jumps in its steady state value with respect to transition firing rates variations (i.e. the steady state throughput, as a function of the transition firing rates is not a continuous function). Moreover, [Meyer, 2012] explained these throughput jumps via *discontinuity induced bifurcations*; connection between the topology of the net and those jumps, however, is not addressed in that work.

The connection of bifurcations and PN structural properties is addressed in this work. It focuses on strongly connected nets in which, under some conditions based on its structure, a bifurcation scenario is characterized. It is proved that for *join free* (JF), *choice free* (CF) and *topologically equal conflict* (TEQ) net systems, structural properties guarantee that bifurcations are not possible. For *mono-T-semiflow* (MTS) net systems a necessary condition is derived for them to exhibit bifurcations. Furthermore, a relationship between this scenario and non-monotonic throughput of the system is presented.

This paper is organized as follows, in section 2 basic concepts on Continuous Petri nets (CPN) and Timed CPN (TCPN) are defined. Section 3 addresses strongly connected and conservative JF-TCPN, and strongly connected and consistent CF-TCPN systems showing that in these classes of TCPN bifurcations are not possible. Section 4 presents a necessary condition for Mono-T-

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Semiflow TCPN systems to exhibit discontinuity-induced bifurcations. Also, this section relates the throughput non monotonicity with bifurcations in Mono-T-Semiflow TCPN. Section 5 presents some concluding remarks.

2. BASIC CONCEPTS

We assume that the reader is familiar with discrete Petri nets, in this section we will give a definition of *continuous Petri nets* (CPN) and *timed continuous Petri nets* (TCPN).

Definition 2.1. A continuous Petri net system is a pair $(N, \boldsymbol{m}_0)$ where $N = (P, T, \boldsymbol{Pre}, \boldsymbol{Post})$ is a Petri net structure with a set of places P, a set of transitions T and the pre and post incidence matrices $\boldsymbol{Pre}$ and $\boldsymbol{Post}$ and the initial marking $\boldsymbol{m}_0$.

The set of input and output nodes of v is denoted as ${}^\bullet v$ and $v^\bullet$, respectively. Each place p_i has a marking denoted by $m_i \in \{\mathbb{R}^+ \cup 0\}$. A transition $t_j \in T$ is enabled at $\boldsymbol{m}$ if $\forall p_i \in {}^\bullet t_j$, $m_i > 0$ and its enabling degree is

$$enab(t_j, \boldsymbol{m}) = \min_{p \in {}^\bullet t_j} \left\{ \frac{\boldsymbol{m}[p]}{\boldsymbol{Pre}(p, t_j)} \right\}$$

Firing of transition t in a certain real ammount $0 \le \alpha \le enab(t, \boldsymbol{m})$ leads to a new marking $\boldsymbol{m}' = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}(t)$ where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token-flow matrix and $\boldsymbol{C}(t)$ is the column of $\boldsymbol{C}$ associated to transition t. If $\boldsymbol{m}$ is reachable from $\boldsymbol{m}_0$ through a sequence σ a fundamental equation can be written $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C\sigma}$, where $\boldsymbol{\sigma} \in (\mathbb{R}^+ \cup \{0\})^{|T|}$ is the firing count vector.

As in discrete Petri nets, a CPN is *bounded* if for every reachable marking $\boldsymbol{m}$ there exists a constant b_m such that for all $p \in P$, $m_i \le b_m$. It is *live* when every transition is *live* (it can ultimately occur for every reachable marking). Reachability may be extended to *lim-reachability* assuming that infinitely long sequences can be fired. The set of all reachable markings at the limit is denoted by lim-RS. A transition t is *lim-live* if no sequence of successively reachable markings exists which converges to a marking such that none of it successors enables t.

A net N is *structurally bounded* if for every initial marking $\boldsymbol{m}_0$, the system $(N, \boldsymbol{m}_0)$ is bounded and *structurally live* when an $\boldsymbol{m}_0$ exists such that $(N, \boldsymbol{m}_0)$ is live.

A column vector $\boldsymbol{y}$ is called a P-flow if $\boldsymbol{y}^T \cdot \boldsymbol{C} = \boldsymbol{0}$; if $\boldsymbol{y} \ge \boldsymbol{0}$ it is called a P-semiflow. Similarly, a column vector $\boldsymbol{x}$ is called a T-flow if $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$; if $\boldsymbol{x} \ge \boldsymbol{0}$ it is called a T-semiflow. In this work, we always consider net systems whose initial marking $\boldsymbol{m}_0$ marks all P-semiflows. Matrix $\boldsymbol{B}_y$ ($\boldsymbol{B}_x$) denotes a basis of P-flows (T-flows). We say that a net N is *conservative* (Cv) when $\boldsymbol{y} > \boldsymbol{0}$, $\boldsymbol{y}^T \cdot \boldsymbol{C} = \boldsymbol{0}$ and it is *consistent* (Ct) when $\boldsymbol{x} > \boldsymbol{0}$, $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$. The support of a vector $\boldsymbol{v}$ is denoted by $\|\boldsymbol{v}\|$ and is the set of indices corresponding to the non-zero values. We say that two transitions t and t' are in *conflict* if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$. A conflict $\{t, t'\}$ is said to be *topologically equal* (TEQ) if $\exists \gamma > 0$ such that $\boldsymbol{Pre}(P, t) = \gamma \boldsymbol{Pre}(P, t')$.

Through this work we assume that N is strongly connected. This is quite reasonable in practice since a necessary condition for a net to be structurally live and structurally bounded is to be consistent and conservative. Furthermore, if the net is connected consistent and conservative, then it is strongly connected [Memmi and Roucairol, 1980].

Petri nets can be classified according to their structure

Definition 2.2. Let N be a Petri net

- N is Choice Free (CF) if $\forall p \in P$, $|p^\bullet| \le 1$
- N is Join Free (JF) if $\forall t \in T$, $|{}^\bullet t| \le 1$
- N is Topologically Equal Conflict (TEQ) if all the conflicts are topologically equal.

Note that if N is strongly connected then for CF nets $|p^\bullet| = 1$. Similarly, for JF nets $|{}^\bullet t| = 1$.

Definition 2.3. A timed continuous Petri net (TCPN) system is a CPN system together with a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^{|T|}$, where its i-th element λ_i is the firing rate associated to the transition t_i.

In this way the fundamental equation depends on time: $\boldsymbol{m}(\tau) = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$, and its time derivative results in the equation $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. The derivative of the firing sequence is the *flow* of the timed model $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$

Under the *infinite servers semantics* this flow is defined as

$$\boldsymbol{f}[t_j] = \boldsymbol{\lambda}[t_j] \cdot \min_{p \in {}^\bullet t_j} \left\{ \frac{\boldsymbol{m}[p]}{\boldsymbol{Pre}(p, t_j)} \right\} \tag{1}$$

for each transition t_j. The *min* operator leads to the concept of configurations. It will be said that the arc (p, t) restricts the dynamic of transition t at $\boldsymbol{m}$ if $\boldsymbol{f}[t] = \boldsymbol{\lambda}[t] \cdot \boldsymbol{m}[p]/\boldsymbol{Pre}[p, t]$. A configuration $\mathcal{C}$ at a marking $\boldsymbol{m}$ is a set of (p, t) arcs, one per transition, restricting the dynamics of the TCPN system. In this work, the *T-coverture*, $\mathcal{T}$, represents the places associated to the configuration. The number of configurations depends on the net structure, it is $\prod_{t \in T} |{}^\bullet t|$. For each configuration we can associate a matrix Π as follows

$$\Pi(\boldsymbol{m})_{i,j} = \begin{cases} \dfrac{1}{\boldsymbol{Pre}(p_j, t_i)} & \text{if } (p_j, t_i) \text{ is constraining } t_i \\ 0 & \text{otherwise} \end{cases}$$

If at a given marking more than one arc is constraining a transition any of them can be used, but only one is taken. Note that each element of vector $\Pi(\boldsymbol{m})\boldsymbol{m}$ is the enabling degree of each transition and that $\boldsymbol{f} = \Lambda \Pi(\boldsymbol{m}) \boldsymbol{m}$ is a vector with the flow of them, where $\Lambda = diag(\lambda_1, \ldots, \lambda_{|T|})$.

The set of all non-negative markings that agree with the P-flows is denoted as $Class(\boldsymbol{m}_0) = \{\boldsymbol{m} \ge \boldsymbol{0} \mid \boldsymbol{B}_y^T \boldsymbol{m} = \boldsymbol{B}_y^T \boldsymbol{m}_0\}$; so, any reachable marking belongs to it. This set can be divided into *marking regions* according to the *configurations*. If N is consistent, a marking region is defined as the set $\mathcal{R}_i = \{\boldsymbol{m} \in Class(\boldsymbol{m}_0) \mid \Pi_i \boldsymbol{m} \le \Pi_j \boldsymbol{m}, \forall \Pi_j\}$. Thus, to each configuration $\mathcal{C}_i$ it is associated a matrix Π_i and a region $\mathcal{R}_i$. These regions are polyhedrons, and are disjoint, except on the borders.

The evolution of the marking of the system is governed by its state equation $\dot{\boldsymbol{m}} = \boldsymbol{C} \Lambda \Pi(\boldsymbol{m}) \boldsymbol{m}$ which is a dynamical piecewise linear system. For each region $\mathcal{R}_i$ one linear time invariant system drives the evolution of the marking, its associated dynamic matrix is $A_i = \boldsymbol{C} \Lambda \Pi_i$.