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Cause versus Effect in Hybrid Systems: A Rigorous Non-standard Analysis Approach

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Abstract: A new approach for modeling nonlinear impulsive system is suggested based on nonstandard analysis. Basic properties of the hyperreals in nonstandard analysis are revisited, and extended to define generalized functions via a sequence approach. The extended generalized functions yield a non unique definition for a Heaviside function and the delta function, which are used to characterize a nonsmooth vector field. By using these extended generalized function, a causal way for characterizing jumps in discontinuous systems follows. An example for a simple affine system illustrates the usefulness of this theoretical development.

Keywords: impulsive system, nonlinear impulsive system, nonstandard analysis.

1. INTRODUCTION

Impulsive systems are a subclass of hybrid systems where the dynamics of motions are modeled with a continuous vector field in the presence of jumps. Earlier work on impulsive differential equations and their dynamics can be found in Lakshmikantham et al. (1989), Zavalishchin and Sesekin (1997) and Yang (2001). These works have successfully build a theory with formal definitions and initial assumptions on the impulsive systems. In addition, fundamental analysis on the system properties of the impulsive systems was thoroughly analyzed in the book, Haddad et al. (2014). Beyond the theory, it extends to the applications in biological system and ecological system in Grognard (2014),Verriest (2003) and Verriest and Pepe (2009). Furthermore, a numerical approach on solving ODEs with discontinuous vector fields can be found in Dieci and Lopez (2012) and Dieci and Guglielmi (2013).

Generally, there are two types of impulsive system; one with the time of jumping events specified and the other where switching times are implicitly determined by additional state dependent dynamics. By using the classical definition in Lakshmikantham et al. (1989), the impulsive dynamics is an interaction between the continuous dynamics and the state jumps. The formulation of the impulsive system equation for a scalar case are given as

$$\dot{x}(t) = f(x(t), u(t)) \qquad \text{if } t \in \mathcal{T}\backslash\bigcup_{i=0}^{\infty} t_i \tag{1}$$

$$x(t_i+) = g(x(t_i-), u(t_i-)) \;\; \text{if } t \in \bigcup_{i=0}^{\infty} t_i. \tag{2}$$

The functions f and g are a continuous functions from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ where the domain is $\mathcal{T} \subset \mathbb{R}$. The sequence $\{t_i\}_i$ are times for resetting the state, and $x(t_i+) := \lim_{t \overset{>}{\to} t_i} x(t)$ and $x(t_i-) := \lim_{t \overset{<}{\to} t_i} x(t)$ are the right and left limit points at t_i.

The above equations successfully describe how the instantaneous jump should be modeled. However, this classical formulation requires a predefined jump behavior, and so should be called an *effect* model, since the dynamics are described without modeling the *cause* of the instantaneous changes.

An exceptional example is when $f(x, u) := Ax + Bu$ is a linear function and $g(x(t_i-), u(t_i-)) := g_i + x(t_i-)$ for some constants A, B and g_i. If (A, B) is reachable, no additional structure is needed as the g_i may be generated by general impulsive input u fed through the read-in matrix B. The above equation can now be modeled without the state jump equation by using the weighted delayed impulse train, as a causation.

$$\dot{x}(t) = Ax(t) + B\sum_{k=1}^{\infty}\sum_{i=0}^{n-1} u_{i,k}\delta^{(i)}(t - t_k) \tag{3}$$

where $\{u_{i,k}\}$ are corresponding control constants satisfying the effect equation and $\delta^{(i)}$s are the $i-$th derivatives for delta. This example displays that the singular control can be used to design a *cause* of the instantaneous changes; here we call such a model a *causal* model.

Therefore, to have a full control on the behavior of the effect in more general cases, there is an urge for extending the usage of singular control to the nonlinear cases. However, applying the singular control in nonlinear systems encounters a critical problem since the singular function defined by Schartz distribution have limitations. The powers of δ and the multiplication of δ with a non-smooth function are ill-defined in the Schartz distribution. More details can be found on Gelfand and Shilov (1969).

The deficiency in the Schwartz distribution theory have been overcame by the new generalized function given by Colombeau (2000) and Colombeau (2011), which have rigorously defined an algebraic structure on the set of distributions. In Todorov and Vernaeve (2008), it is shown that the theory of Colombeau algebra can also be interpreted in

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the language of the non standard analysis(NSA) which was first introduced in Robinson (1967). The application of this generalized function to the ordinary differential equation can be found in Colombeau (2011), and Kunzinger and Steinbauer (1998) have adapted the Colombeau algebra to an impulsive gravitational wave equation. In this paper, we also suggest a framework to define the generalized function and its multiplications using the extended time line in NSA.

Another approach to define a causal model for nonlinear impulsive systems is to regard the singular control as a sequence of functions. Miller (1996) and Orlov (1997) have independently shown the existence of a causal model for a nonlinear impulsive system by defining the singular function as a sequence of integrable functions, $\{u_k\}$, for which its state solution converges to the effect equation, Eqn 2, in the weak$*$ topology. The paper suggests an auxiliary system for the new state w, where $\int_0^1 w(t)dt$ equals to the jump in Eqn 2, in order to define the sequence of functions $\{u_k\}$ uniquely. Further research based on this formulation can be found in the books in Miller and Rubinovich (2003) and Orlov (2008). In addition, a sequence based causal model for nonlinear systems has also been studied in Verriest (2014), Bressan Jr and Rampazzo (1991) and Aronna and Rampazzo (2014). Especially, Verriest (2014) briefly showed a connection to the NSA, which will be formally defined and extended to the space of generalized functions in this paper. See also Verriest (1990).

In the following sections, the fundamental definitions in NSA are recapped, and a framework to continuize the discontinuous functions is suggested. By using the continuization, a new definition of a piecewise continuous function and its derivatives, such as a δ function, is provided as the extended function in NSA. Next, we show that the multiplication of these functions is well defined and closed in the newly defined space of functions. Finally, we apply this framework to design a *causal* model for the nonlinear impulsive system especially for the affine system.

2. PRELIMINARY

In this preliminary section, we define the basic operators used throughout the paper, and summarize the fundamental definitions in NSA. See Goldblatt (1998) for details.

2.1 Basic Operators

Let $C(\mathbb{R})$ be a set of continuous function in $\mathbb{R}$ and $\alpha \in \mathbb{R}$ be a fixed constant.

Definition 1. (Evaluation operator). A functional, $\sigma_t : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ is called an evaluation operator at time $t \in \mathbb{R}$ if $\sigma_t(x) = x(t)$ for an arbitrary function $x \in \mathbb{R}^{\mathbb{R}}$.

By using the evaluation operator, we define the translation and the scaling operator acting on a continuous functions.

Definition 2. (Translation operator and scaling operator). Operators, $T_\alpha : C(\mathbb{R}) \to C(\mathbb{R})$ and $S_\alpha : C(\mathbb{R}) \to C(\mathbb{R})$, are called translation and scaling operators by the factor α, respectively, if $\sigma_t(T_\alpha(x)) = \sigma_{t+\alpha}(x)$ and $\sigma_t(S_\alpha(x)) = \sigma_{\alpha t}(x)$ for $\forall x \in C(\mathbb{R})$ and $\forall t \in \mathbb{R}$.

The next lemma shows the commutation of these operators. Fix $\alpha, \beta \in \mathbb{R}$ to be a constant.

Lemma 3. (a) $S_\alpha \circ S_\beta = S_\beta \circ S_\alpha = S_{\alpha\beta}$
(b) $T_\alpha \circ T_\beta = T_\beta \circ T_\alpha = T_{\alpha+\beta}$
(c) $S_\alpha \circ T_\beta = T_{\frac{\beta}{\alpha}} \circ S_\alpha$
(d) $D \circ S_\alpha = \alpha S_\alpha \circ D$

Proof. The proofs are immediate from the definition.

In addition, throughout this paper, we use a new notation for a geometric sequence. Suppose $\{s \cdot r^n\}$ is a geometric sequence with initial value s and the rate r, then $\langle r, s\rangle := \{s \cdot r^n\}$.

2.2 Nonstandard Analysis

NSA is motivated from the construction of the reals, $\mathbb{R}$, from the rational numbers, $\mathbb{Q}$, by taking the equivalence classes of the space of the Cauchy sequences with rational numbers. Similarly, the first objective of NSA is to give a proper extension to the $\mathbb{R}$ space with the set of real-valued sequences. Let $\mathbb{R}^{\mathbb{N}}$ be the set of real-valued sequences, and $\mathcal{P}(\mathbb{N})$ be the power set on $\mathbb{N}$. A *filter* on $\mathbb{R}^{\mathbb{N}}$ is a nonempty collection of $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ which satisfy the first two axioms.

- If $A, B \subset \mathcal{F}$, then $A \cap B \in \mathcal{F}$
- If $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$, then $B \in \mathcal{F}$
- For any $A \subset \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$

An *ultrafilter* is a proper filter which also satisfies the last axiom. By using the Corollary 2.6.2 in Goldblatt (1998), it can be shown that the equivalence relation $\cong$ on $\mathbb{R}^{\mathbb{N}}$,

$$\langle r_n\rangle \cong \langle s_n\rangle \text{ iff}\{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

is well defined where F is the ultrafilter on $\mathbb{N}$, and $\{r_n\}$ and $\{s_n\}$ are in $\mathbb{R}^{\mathbb{N}}$. Let the equivalence class of a sequence $\{r_n\} \in \mathbb{R}^{\mathbb{N}}$ be denoted as $\langle r\rangle$.

Definition 4. The quotient set ${}^*\mathbb{R} := \{\langle r\rangle : \{r_n\} \in \mathbb{R}^{\mathbb{N}}\}$ is called the extended real space or *hyperreal space*, and the members of ${}^*\mathbb{R}$ are called a *hyperreal number.*

Since $\mathbb{R}^{\mathbb{N}}$ can have a sequence with repeated elements, the real space $\mathbb{R}$ is a proper subspace of ${}^*\mathbb{R}$. Further, ${}^*\mathbb{R}$ is endowed with algebraic structure by defining addition and multiplication by

$$\langle r\rangle + \langle s\rangle = \langle\{r_n + s_n\}\rangle \tag{4}$$

$$\langle r\rangle \cdot \langle s\rangle = \langle\{r_n \cdot s_n\}\rangle \tag{5}$$

and an order relation, $<$, by

$$\langle r\rangle < \langle s\rangle \text{ iff } \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}. \tag{6}$$

Theorem 5. (Hyperreal space as a ordered field). The hyperreal space, $\langle {}^*\mathbb{R}, +, \cdot, <\rangle$, is an ordered field with zero $\langle 0\rangle$ and unity $\langle 1\rangle$

Proof. See Theorem 3.6.1 in Goldblatt (1998).

One of the strong benefits of having an extended real space is that now there exist elements which are infinitesimally small and also there exist unbounded numbers.

Definition 6. (Infinitesimal and unlimited number). An element $\langle\{\epsilon_n\}\rangle \in {}^*\mathbb{R}$ is called *infinitesimal* if $A = \{n \in \mathbb{N} :$