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Robustness of recurrence for a class of stochastic hybrid systems ⋆

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Abstract: We study a weak stability property called recurrence for a class of stochastic hybrid systems. Robustness of the recurrence property to various state-dependent perturbations is established under mild regularity conditions for the stochastic hybrid system. A potential application of the robustness results in the development of converse Lyapunov theorems is also outlined.

Keywords: Recurrence, Robustness, Stochastic hybrid systems

1. INTRODUCTION

Hybrid systems are a class of dynamical systems that involve both continuous-time evolution and discrete-time events. Stochastic hybrid systems (SHS) generalize this class by adding randomness. In stochastic hybrid systems randomness can affect the continuous-time dynamics, discrete-time dynamics and also the transition between them. Frameworks for modeling SHS are in Yin and Zhu (2009),Teel (2013), Davis (1984), Bujorianu and Lygeros (2006) and Teel et al. (2014b).

Modeling frameworks for stochastic hybrid systems seldom account for systems that generate non-unique solutions. In Teel (2013) and Teel (2014b) a generalized framework for modeling SHS with non-unique solutions is proposed. The need to study stochastic systems with non-unique solutions is twofold. Firstly, analyzing such systems is crucial to developing a robust stability theory. Robust stability for non-stochastic hybrid systems is studied in Goebel et al. (2012). Secondly, such system models allow flexibility in control design applications. See Subbaraman et al. (2013), Venkateswaran et al. (2013) and Poveda et al. (2015).

The stochastic stability property of interest in this paper is called recurrence. Loosely speaking, recurrence of an open set implies that solutions visit the set infinitely often with probability one. The recurrence property is frequently studied in the literature. See Yin and Zhu (2009) and Meyn and Tweedie (2009). Recurrence is a weaker notion of stability compared to more commonly studied notions of mean square asymptotic stability and asymptotic stability in probability. Recurrence of an open bounded set need not imply any stability-like property for the set. Recurrence of a set does not imply any invariance-like property for the set and finally, recurrence of an open, bounded set does not imply that solutions stay bounded in a probabilistic sense. Recurrence is still a useful property to study since it provides an alternative when stronger properties like convergence or asymptotic stability are impossible to establish. This is particularly true in the case of systems affected by persistent disturbances. We refer the reader to (Subbaraman and Teel, 2015, Example 1) and Isaacs et al. (2014) for more details.

Robustness of stability can be loosely defined as the stability property being preserved when the nominal system is affected by sufficiently small perturbations. The robustness results in this paper exploit the good regularity properties of the nominal system. The results generated in this work have potential application in establishing novel converse Lyapunov theorems for stochastic hybrid systems. For non-stochastic hybrid systems, results on robustness of the recurrence property and the relation between recurrence and uniform ultimate boundedness of solutions are presented in Subbaraman and Teel (2015).

The rest of the paper is organized as follows. Section 2 presents the basic notation and definitions to be used in the paper. Section 3 introduces the SHS framework that will be considered in the rest of the paper. The recurrence property is explained in Section 4. Section 5 introduces viability and reachability probabilities which can be used to prove the main results of the paper. Section 6 establishes some basic bounds related to these probabilities. The main results are presented in Section 7. Section 8 establishes the link between the main results and the development of converse Lyapunov theorems. Section 9 presents some concluding comments and future work. The proofs of the main results are not presented due to space restrictions.

2. BASIC NOTATION AND DEFINITIONS

For a closed set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$ is the Euclidean distance of x to S. $\mathbb{B}$ (resp., $\mathbb{B}^o$) denotes the closed (resp., open) unit ball in $\mathbb{R}^n$. Given a closed set $S \subset \mathbb{R}^n$ and $\epsilon > 0$, $S + \epsilon\mathbb{B}$ represents the set $\{x \in \mathbb{R}^n : |x|_S \leq \epsilon\}$. $\mathbb{R}_{\geq 0}$ denotes the non-negative real numbers; $\mathbb{Z}_{\geq 0}$ denotes the non-negative integers. Let $\mathcal{T}$ be a topological space. A function $\Psi : \mathcal{T} \to \mathbb{R}_{\geq 0}$ is *upper semicontinuous* if for every converging sequence $\{t_i\} \to t$,

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$\limsup_{i\to\infty} \Psi(t_i) \leq \Psi(t)$. For $S \subset \mathbb{R}^n$, the symbol $\mathbb{I}_S$ denotes the indicator function of S i.e., $\mathbb{I}_S(x) = 1$ for $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. Following Teel (2013), we define for sets $S_1, S_2 \subset \mathbb{R}^p$, $\mathbb{I}_{\subset S_1}(S_2) = 1 - \max_{x\in S_2} \mathbb{I}_{\mathbb{R}^n\setminus S_1}(x)$ and $\mathbb{I}_{\cap S_1}(S_2) = \max_{x\in S_2} \mathbb{I}_{S_1}(x)$ with the convention that the maxima are zero when $S_2 = \varnothing$. For $\tau \geq 0$, we define the set $\Gamma_{\leq\tau} := \{(s,t) \in \mathbb{R}^2 : s+t \leq \tau\}$. (The sets $\Gamma_{<\tau}, \Gamma_{\geq\tau}$ are defined similarly). A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if, for each $(x_i, y_i) \to (x, y) \in \mathbb{R}^p \times \mathbb{R}^n$ satisfying $y_i \in M(x_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, $y \in M(x)$. A mapping M is *locally bounded* if, for each bounded set $K \subset \mathbb{R}^p$, $M(K) := \bigcup_{x\in K} M(x)$ is bounded. $\mathbf{B}(\mathbb{R}^m)$ denotes the Borel σ-field. A set $F \subset \mathbb{R}^m$ is measurable if $F \in \mathbf{B}(\mathbb{R}^m)$. For a measurable space $(\Omega, \mathcal{F})$, a mapping $M : \Omega \rightrightarrows \mathbb{R}^n$ is *measurable* (Rockafellar and Wets, 1998, Def. 14.1) if, for each open set $\mathcal{O} \subset \mathbb{R}^n$, the set $M^{-1}(\mathcal{O}) := \{\omega \in \Omega : M(\omega) \cap \mathcal{O} \neq \varnothing\} \in \mathcal{F}$. The functions $\pi_i : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are such that $\pi_i(t_1, t_2, z) = t_i$ for each $i \in \{1, 2\}$.

3. PRELIMINARIES ON STOCHASTIC HYBRID SYSTEMS

We consider a class of SHS with randomness restricted to the discrete-time dynamics introduced in Teel (2013). Let the state $x \in \mathbb{R}^n$ and random input $v \in \mathbb{R}^m$. Then, the SHS is written formally as

$$x \in C \qquad \dot{x} \in F(x) \tag{1}$$

$$x \in D \qquad x^+ \in G(x, v^+) \tag{2}$$

$$\mu(\cdot) \tag{3}$$

where $C, D \subset \mathbb{R}^n$ represent the flow and jump sets respectively and F, G represent the flow and jump maps respectively. So, the continuous-time dynamics is modeled by a differential inclusion and the discrete-time dynamics is modeled by a stochastic difference inclusion.

The distribution function μ is derived from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent, identically distributed (i.i.d.) input random variables $\mathbf{v}_i : \Omega \to \mathbb{R}^m$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ for $i \in \mathbb{Z}_{\geq 1}$. Then μ is defined as $\mu(A) = \mathbb{P}(\omega \in \Omega : \mathbf{v}_i(\omega) \in A)$ for every $A \in \mathbf{B}(\mathbb{R}^m)$. We denote by $\mathcal{F}_i$ the collection of sets $\{\omega : (\mathbf{v}_1(\omega), ..., \mathbf{v}_i(\omega)) \in A\}$, $A \in \mathbf{B}((\mathbb{R}^m)^i)$ which are the sub-σ fields of $\mathcal{F}$ that form the natural filtration of $\mathbf{v} = \{\mathbf{v}_i\}_{i=1}^{\infty}$. For simplicity we will refer to the data of the stochastic hybrid system through its data as $\mathcal{H} = (C, F, D, G, \mu)$. We now define the notion of random solution to $\mathcal{H}$ under the following basic assumptions on the system data which is imposed throughout the rest of the paper.

Standing Assumption 1. The data of the SHS $\mathcal{H}$ satisfies the following conditions:

(1) The sets $C, D \subset \mathbb{R}^n$ are closed;
(2) The mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer-semicontinuous, locally bounded with nonempty convex values on C;
(3) The mapping $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded and the mapping $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^{2n} : y \in G(x, v)\}$ is measurable with closed values.

The need for the regularity properties listed in Standing Assumption 1 are twofold. Firstly the robustness results established in this paper exploit the system properties in Standing Assumption 1. Secondly, it guarantees the existence of random solutions for the SHS. See Teel (2013) for more details.

We first explain the notion of a solution to non-stochastic hybrid systems studied in Goebel et al. (2009) and Goebel et al. (2012). A compact hybrid time domain is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ of the form $\cup_{j=0}^{J}([t_j, t_{j+1}] \times \{j\})$ for some $J \in \mathbb{Z}_{\geq 0}$ and real numbers $0 = t_0 \leq t_1 .... \leq t_{J+1}$. A hybrid time domain is a set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that for each T, J, the set $E \cap ([0, T] \times \{0, ..., J\})$ is a compact hybrid time domain. A hybrid arc is a mapping $\phi : E \to \mathbb{R}^n$ such that E is a hybrid time domain and for each $j \in \mathbb{Z}_{\geq 0}$, the mapping $t \mapsto \phi(t, j)$ is locally absolutely continuous.

Let $(\Omega, \mathcal{F})$ be a measurable space. A stochastic hybrid arc is a mapping $\mathbf{x}$ defined on Ω such that $\mathbf{x}(\omega)$ is a hybrid arc for each $\omega \in \Omega$ and the set-valued mapping from Ω to $\mathbb{R}^{n+2}$ defined by

$$\omega \mapsto \text{graph}(\mathbf{x}(\omega)) := \{(t, j, z) : \phi = \mathbf{x}(\omega), (t, j) \in \text{dom}(\phi), z = \phi(t, j)\}$$

is $\mathcal{F}$-measurable with closed values. Define $\text{graph}(\mathbf{x}(\omega))_{\leq j} := \text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \{0, ..., j\} \times \mathbb{R}^n)$. An $\{\mathcal{F}_j\}_{j=0}^{\infty}$ adapted stochastic hybrid arc is a stochastic hybrid arc $\mathbf{x}$ such that the mapping $\omega \mapsto \text{graph}(\mathbf{x}(\omega))_{\leq j}$ is $\mathcal{F}_j$ measurable for each $j \in \mathbb{Z}_{\geq 0}$. An adapted stochastic hybrid arc $\mathbf{x}$ is a solution starting from x denoted $\mathbf{x} \in \mathcal{S}_r(x)$ if $\mathbf{x}(\omega)$ is a solution to $\mathcal{H}$ with inputs $\{\mathbf{v}_i(\omega)\}_{i=1}^{\infty}$; that is with $\phi_\omega := \mathbf{x}(\omega)$ we have

- $\phi_\omega(0, 0) = x$;
- if $(t_1, j), (t_2, j) \in \text{dom}(\phi_\omega)$ with $t_1 < t_2$ then, for almost every $t \in [t_1, t_2]$, $\phi_\omega(t, j) \in C$ and $\dot{\phi}_\omega(t, j) \in F(\phi_\omega(t, j))$;
- if $(t, j), (t, j+1) \in \text{dom}(\phi_\omega)$ then $\phi_\omega(t, j) \in D$ and $\phi_\omega(t, j+1) \in G(\phi_\omega(t, j), \mathbf{v}_{j+1}(\omega))$.

We refer the reader to Teel (2013) for more details on the solution concept to SHS. In this paper, we will sometimes focus on solutions that are maximal (See (Goebel et al., 2012, Def 2.7)).

4. RECURRENCE AND UNIFORM RECURRENCE

In this section we define the notion of recurrence for open sets. An open, bounded set $\mathcal{O} \subset \mathbb{R}^n$ is *globally recurrent* for $\mathcal{H}$ if there are no finite escape times for (1) and for each $x \in \mathbb{R}^n$ and $\mathbf{x} \in \mathcal{S}_r(x)$,

$$\lim_{\tau\to\infty} \mathbb{P}\Big(\big(\text{graph}(\mathbf{x}) \subset (\Gamma_{<\tau} \times \mathbb{R}^n)\big) \vee \big(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq\tau} \times \mathcal{O})\big) \Big) = 1.$$

Loosely speaking, the above condition insists that almost surely the sample paths of the random solution $\mathbf{x}$ are either not complete or hit the set $\mathcal{O}$. We refer the reader to Teel (2013) for more details. An open, bounded set $\mathcal{O} \subset \mathbb{R}^n$ is *uniformly globally recurrent* for $\mathcal{H}$ if there are no finite escape times for (1) and for each $\rho > 0$ and $R > 0$ there exists $\tau \geq 0$ such that for $\xi \in R\mathbb{B}$ and $\mathbf{x} \in \mathcal{S}_r(\xi)$,

$$\mathbb{P}\Big(\big(\text{graph}(\mathbf{x}) \subset (\Gamma_{<\tau} \times \mathbb{R}^n)\big) \vee \big(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq\tau} \times \mathcal{O})\big) \Big)$$