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IFAC-PapersOnLine 48-27 (2015) 358–363

Stability Verification of Nearly Periodic Impulsive Linear Systems using Reachability Analysis ⋆

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Abstract:
The paper provides stability analysis to certain classes of hybrid systems, more precisely impulsive linear systems. This analysis is conducted using the notion of reachable set. The main contribution in this work is the derivation of theoretical necessary and sufficient conditions for impulsive linear systems with nearly periodic resets subject to timing contracts. This characterization serves as the basis of a computational method for the stability verification of the considered class of systems. In addition, we show how this work handles the problem of timing contract synthesis for the considered class and we generalize our approach to verify stability of impulsive linear systems with stochastic reset instants. Applications on sampled-data control systems and comparisons with existing results are then discussed, showing the effectiveness of our approach.

Keywords: Reachability; Impulsive systems; Stability analysis.

1. INTRODUCTION

Impulsive dynamical systems form a class of hybrid systems which models processes that evolve continuously and undergo instantaneous changes at discrete time instants. Applications of impulsive dynamical systems include sampled-data control systems [Briat (2013)] or networked control systems [Donkers et al. (2011)]. In this paper, we consider the problem of verifying stability of impulsive linear systems subject to nearly periodic resets. More precisely, the duration between two consecutive resets is uncertain but constrained in some bounded interval given by a timing contract. Several approaches are developed in literature to analyze stability of such systems. On one hand, there is a discrete-time and convex embedding approach [Hetel et al. (2011, 2013)], a time delay technique [Liu et al. (2010); Seuret and Peet (2013)], a hybrid system formulation [Dai et al. (2010)], and an Input/Output stability approach [Omran et al. (2014); Fujioka (2009)]. On the other hand, [Fiacchini and Morarescu (2014)] derives an approach which mainly uses backward invariant set computations [Blanchini and Miani (2007)] to find a contracting polytopic set for the system. In the former methods, the stability criterion is given in terms of Linear Matrix Inequality (LMI) which numerically provides only sufficient conditions for stability. Whereas, the latter is less conservative but in its turn does provide only sufficiency for the stability verification problem, knowing that computational-wise necessity would come with possibly unbounded computations.

In this paper, we propose a new approach based on forward reachability analysis. Primarily, we state the necessary and sufficient theoretical conditions, based on reachable sets, for stability of NPILS. Then, we take advantage of previous work[Le Guernic and Girard (2010)], which provides an algorithmic scheme to compute the reachable sets for linear systems avoiding the wrapping effect (accumulation of over-approximation errors). We present a computational reachability based stability verification method for nearly periodic impulsive linear systems. Moreover, we handle the problem of synthesizing timing contracts using the previous method as well as monotonicity of the stability property with respect to parameters of the timing contract. The work is then extended to deal with the problem of stability verification in the stochastic case. Last, the efficiency of our work is shown by illustrative examples where our results are not only less conservative than several of those existing in literature but also show tightness of our approximation scheme. Advantages in using our method are seen to extend in dealing with further timing contracts as well as taking into consideration some performance specifications since an insight on the reachable set is given during our analysis.

The paper is organized as follows. First, some preliminary notations are defined before formulating the stability verification and the timing contract synthesis problems in Section 3. The main results are discussed in Section 4. Detailed explanations on the algorithms and over-approximation scheme utilized in our approach lies in Section 5. In Section 6 stability is studied for the stochastic case. We discuss more on the computation of the reachable set in Section 7. Applications on sampled-data control systems and comparisons with existing results are then

⋆ This work was supported by the Agence Nationale de la Recherche (COMPACS project ANR-13-BS03-0004).

10.1016/j.ifacol.2015.11.200

discussed, before concluding our work. Due to space limitation proofs of the theorems, corollaries, and propositions are omitted.

Notations Let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{N}^+$ denote the sets of reals, nonnegative reals, positive reals, nonnegative integers and positive integers, respectively. For $I \subseteq \mathbb{R}_0^+$, let $\mathbb{N}_I = \mathbb{N} \cap I$. Given a real matrix $A \in \mathbb{R}^{n\times n}$, $|A|$ is the matrix whose elements are the absolute values of the elements of A. Given $\mathcal{S} \subseteq \mathbb{R}^n$ and a real matrix $A \in \mathbb{R}^{n\times n}$, the set $A\mathcal{S} = \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = Ay)\}$; for $a \in \mathbb{R}$, $a\mathcal{S} = (aI_n)\mathcal{S}$ where I_n is the $n \times n$ identity matrix. The interior of $\mathcal{S}$ is denoted by $\mathrm{int}(\mathcal{S})$. The convex hull of $\mathcal{S}$ is denoted by $\mathrm{ch}(\mathcal{S})$. The interval hull of $\mathcal{S}$ is the smallest interval containing the set $\mathcal{S}$ and is denoted by $\square(\mathcal{S})$. The symmetric interval hull of $\mathcal{S}$ is the smallest symmetric (with respect to 0) interval containing $\mathcal{S}$ and is denoted by $\boxdot(\mathcal{S})$. Given $\mathcal{S}, \mathcal{S}' \subseteq \mathbb{R}^n$, the Minkowski sum of $\mathcal{S}$ and $\mathcal{S}'$ is $\mathcal{S} \oplus \mathcal{S}' = \{x + x' : x \in \mathcal{S}, x' \in \mathcal{S}'\}$. A polytope $\mathcal{P}$ is a subset of $\mathbb{R}^n$ which can be defined as the intersection of a finite number of closed half-spaces, that is $\mathcal{P} = \{x \in \mathbb{R}^n : Hx \le b\}$ where $H \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and the vector of inequalities is interpreted componentwise. Let H_i, $i \in \mathbb{N}_{[1,m]}$ denote the row vectors of H, and if $0 \in \mathrm{int}(\mathrm{ch}(\{H_1, \dots, H_m\}))$, then $\mathcal{P}$ is compact. Given a template matrix $H \in \mathbb{R}^{m\times n}$ and a compact set $\mathcal{S} \subseteq \mathbb{R}^n$, let us define the polytope $\Gamma_H(\mathcal{S}) = \{x \in \mathbb{R}^n : Hx \le b\}$ where $b_i = \max_{x\in\mathcal{S}} H_i x$, $i \in \mathbb{N}_{[1,m]}$. In other words, $\Gamma_H(\mathcal{S})$ is the smallest polytope whose facets directions are given by H and containing $\mathcal{S}$. We denote the set of all subsets of $\mathbb{R}^n$ by $2^{\mathbb{R}^n}$. We denote by $\mathcal{K}(\mathbb{R}^n)$ the set of compact subsets of $\mathbb{R}^n$ and by $\mathcal{K}_0(\mathbb{R}^n)$ the set of compact subsets of $\mathbb{R}^n$ containing 0 in their interior.

2. PROBLEM FORMULATION

We consider the class of impulsive linear systems given by:

$$\dot{x}(t) = A_c x(t),\ \forall t \in (t_k, t_{k+1}), k \in \mathbb{N}, \tag{1}$$

$$x(t_k^+) = A_d x(t_k),\ k \in \mathbb{N}, \tag{2}$$

where $(t_k)_{k\in\mathbb{N}}$ are the reset instants, $x(t) \in \mathbb{R}^n$ is the state of the system, and $x(t^+) = \lim_{\tau\to 0, \tau>0} x(t+\tau)$.

We assume that the sequence of reset instants (t_k) satisfies a *timing contract* given by

$$t_0 = 0,\ t_{k+1} - t_k = T + \tau_k,\ \tau_k \in [0, \delta],\ k \in \mathbb{N} \tag{3}$$

where $T \in \mathbb{R}^+$ represents a nominal reset period and $(\tau_k)_{k\in\mathbb{N}}$ is a bounded uncertain sequence in the compact set $[0, \delta]$, $\delta \in \mathbb{R}_0^+$. Hybrid dynamical systems described by the continuous dynamics (1), the discrete dynamics (2) and the timing contract (3) are called *nearly periodic impulsive linear systems* (NPILS).

Definition 1. The NPILS (1-3) is *globally uniformly exponentially stable* (GUES) if there exist $\lambda \in \mathbb{R}^+$ and $C \in \mathbb{R}^+$ such that, for all sequences $(t_k)_{k\in\mathbb{N}}$ verifying (3) the solutions of (1-2) verify

$$\|x(t)\| \le Ce^{-\lambda t} \|x(0)\|,\ \forall t \in \mathbb{R}^+.$$

In this paper, we present algorithms for solving the following two problems:

Problem 1. (Stability verification). Given $A_c \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^+$, and $\delta \in \mathbb{R}_0^+$, verify that NPILS (1-3) is GUES.

Problem 2. (Timing contract synthesis). Given $A_c \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, synthesize a set $\Pi \subseteq \mathbb{R}^+ \times \mathbb{R}_0^+$ such that for all $(T, \delta) \in \Pi$, NPILS (1-3) is GUES.

3. STABILITY CHARACTERIZATIONS

This section presents the main theoretical result of the paper in the form of a necessary and sufficient stability condition for NPILS (1-3). This condition can serve to derive a solution to Problem 1. We also prove several results which will be instrumental in solving Problem 2. Before proceeding to the main results, we need to define the notion of reachable set.

Definition 2. Given a continuous-time dynamical system

$$\dot{x}(t) = Ax(t),\ t \in \mathbb{R}_0^+,\ x(t) \in \mathbb{R}^n$$

the *reachable set* on $[t, t'] \subseteq \mathbb{R}_0^+$ from the set $\mathcal{S} \subseteq \mathbb{R}^n$ is

$$\mathcal{R}^A_{[t,t']}(\mathcal{S}) = \bigcup_{\tau\in[t,t']} e^{\tau A}\mathcal{S}.$$

Then, let us introduce the map: $\Phi : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$, defined for all $\mathcal{S} \subseteq \mathbb{R}^n$ by

$$\Phi(\mathcal{S}) = \mathcal{R}^{A_c}_{[0,\delta]}(e^{TA_c} A_d \mathcal{S}). \tag{4}$$

The interpretation of $\Phi(\mathcal{S})$ is as follows. Let x be a trajectory of NPILS (1-3) such that $x(t_k) \in \mathcal{S}$, then $x(t_{k+1}) \in \Phi(\mathcal{S})$. It is easy to see that if $\mathcal{S}$ is compact then so is $\Phi(\mathcal{S})$. It is clear that for two sets $\mathcal{S}$, $\mathcal{S}' \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}$, it holds $\Phi(\mathcal{S} \cup \mathcal{S}') = \Phi(\mathcal{S}) \cup \Phi(\mathcal{S}')$ and $\Phi(a\mathcal{S}) = a\Phi(\mathcal{S})$. We define the iterates of Φ as $\Phi^0(\mathcal{S}) = \mathcal{S}$ for all $\mathcal{S} \subseteq \mathbb{R}^n$, and $\Phi^{k+1} = \Phi \circ \Phi^k$ for all $k \in \mathbb{N}$. Then, it is clear that $x(0) \in \mathcal{S}$ implies that $x(t_k) \in \Phi^k(\mathcal{S})$ for all $k \in \mathbb{N}$.

The exact computation of Φ is often impossible and we use in this work an over-approximation $\overline{\Phi} : \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ satisfying the following assumption:

Assumption 3. For all $\mathcal{S} \in \mathcal{K}(\mathbb{R}^n)$, $\Phi(\mathcal{S}) \subseteq \overline{\Phi}(\mathcal{S})$.

We will discuss an effective computation of $\overline{\Phi}(\mathcal{S})$ in Section 4. The iterates of the map $\overline{\Phi}$ are defined similarly to those of Φ.

The following result characterize the stability of NPILS (1-3) in terms of the map Φ:

Theorem 4. Let $\mathcal{S} \in \mathcal{K}_0(\mathbb{R}^n)$, the following statements are equivalent: [1]

(a) NPILS (1-3) is GUES,
(b) There exists a triplet $(k, j, \rho) \in \mathbb{N}^+ \times \mathbb{N}_{[0,k-1]} \times (0, 1)$ such that $\Phi^k(\mathcal{S}) \subseteq \rho\Phi^j(\mathcal{S})$,
(c) There exists a pair $(k, \rho) \in \mathbb{N}^+ \times (0, 1)$ such that $\Phi^k(\mathcal{S}) \subseteq \rho \bigcup_{j=0}^{k-1} \Phi^j(\mathcal{S})$.

We now derive sufficient conditions for stability based on an over-approximation of map Φ.

[1] Similar results for discrete-time switched systems were shown in [Athanasopoulos and Lazar (2014)].