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Singular optimal control by minimizer flows

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ABSTRACT

This paper studies a computational method to deal with a singular optimal control problem by minimizer flows in a viscosity approximation to the Hamilton–Jacobi–Bellman equation. The boundary of the compact constraint set of control variable is intersected with a class of minimizer flows to yield a Hamiltonian extremal function in rewriting the HJB equation. The analysis properties of the flow are revealed in a global optimization framework. An example on computing a minimizer flow and a Hamiltonian extremal function is presented. An application of the minimizer flow in a non-smooth optimization problem is also mentioned.

1. Primal problem

In this paper we study the following optimal control problem:

$$(\mathcal{P}):\ \min J(0,x_0,u)=Q(x(T))+\int_0^T F(x(t))dt, \tag{1.1}$$

$$s.t.\ \dot{x}=Ax+Bu,\ \ x(0)=x_0,\ t\in[0,T], \tag{1.2}$$

$$x\in R^n,\ u\in U=\{u:p(u)\le 1\}\subset R^m,$$

where $Q(x)$: $R^n\to R^1$, $F(x)$: $R^n\to R^1$ are continuously differentiable and $p(u)$: $R^m\to R^1$ is twice continuously differentiable. We assume that $\nabla^2 p(u)>0,\ \forall u\in R^m$, and

$$\liminf_{\|u\|\to\infty}\frac{p(u)}{\|u\|^2}>0. \tag{1.3}$$

Apparently we need to suppose that U is not empty. In the linear control system (1.2), $A\in R^{n\times n}$, $B\in R^{n\times m}$ are given matrices, and x_0 is a given vector in R^n.

Associated with the problem $(\mathcal{P})$, for the state x and control u with the Lagrange multiplier λ, the Hamiltonian is defined as a function:

$$H(x,u,\lambda)=\lambda^T(Ax+Bu)+F(x). \tag{1.4}$$

Noting that $H_{uu}(x,u,\lambda)\equiv 0$, we see that $(\mathcal{P})$ is a singular optimal control problem([1,8]).

One may note that in general the constraint set U may be unbounded. For example, we can put forth a simple example in R^1 for which $p(u)=e^u$ so that the corresponding set U is unbounded, noting that $\lim_{u\to-\infty}e^u=0$. But with the assumption (1.3) we can show that the constraint set U is bounded and consequently it is compact. For example, if $p(u)=u^Tu$, then the set U is just the unit ball in R^m.

Lemma 1.1. *The constraint set U is compact in R^m.*

Proof. Since $p(u)$ is continuously differentiable, the definition of the set $U=\{u:p(u)\le 1\}$ implies that U is closed. Next we show that U is bounded. It follows from (1.3) that there exist positive numbers M and β such that, when $\|u\|\ge M$, $p(u)\ge\beta\|u\|$. Let $u\in U$. But if $\|u\|\ge M$, then $\|u\|\le\beta^{-1}p(u)\le\beta^{-1}$. Thus, $U\subset\{\|u\|\le\max(M,\beta^{-1})\}$. Consequently U is bounded. The lemma has been proved. □

On the theoretic aspect, it is difficult to find out an optimal control for a nonlinear problem [1,8,9]. Traditionally, in most works on nonlinear optimal control, the authors focus on either numerical viscosity solution to Hamilton–Jacobi–Bellman equations [4,6,10] or theoretical study concerning Pontryagin maximum principle [7,11].

By the classical computational approach to a singular optimal control problem [1], instead of solving the problem $(\mathcal{P})$, the authors consider, for a sufficiently small positive number α, the following optimal control problem:

$$(\mathcal{P}_\alpha):\ \min J(0,x_0,u)=Q(x(T))+\int_0^T\left[F(x)+\frac{\alpha}{2}u^Tu\right]dt \tag{1.5}$$

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$$s.t.\ \dot{x} = Ax + Bu,\ x(0) = x_0,\ t \in [0, T],$$

$$x \in R^n,\ u \in U = \{u : p(u) \le 1\} \subset R^m.$$

In stead of the above approach, this paper presents a computational method to deal with the viscosity solution of the Hamilton–Jacobi–Bellman equation with respect to the singular optimal control problem $(\mathcal{P})$.

Remark 1.1. Let me mention a little bit of the motivation of the approach in this paper. By the traditional way to deal with a singular optimal control, when using the Pontryagin principle, people always meet difficult problems on how to determine the extremal control with respect to the state or co-state variable. In this paper, we try to avoid establishing a relationship between the extremal control and the state or co-state variable. We establish a relationship between an extremal control and value function of the optimal control problem by global optimization method instead.

The rest of the paper is organized as follows. In Section 2, we focus on Hamilton–Jacobi–Bellman equation in a minimization framework. In Section 3, we present a differential flow to make Hamiltonian extremal function corresponding to the Hamilton–Jacobi–Bellman equation with an example to visualize the computing process. In Section 4, we discuss how to compute numerically the viscosity solution to the Hamilton–Jacobi–Bellman equation. An application of the minimizer flow in a non-smooth optimization problem is mentioned in Section 5.

2. Hamilton–Jacobi–Bellman equation

Associated with the optimal control problem $(\mathcal{P})$, the value function is introduced as follows. For $t \in (0, T), x \in R^n$ we consider the problem:

$$\min J(t, x, u) = Q(x(T)) + \int_t^T [F(x)]ds \tag{2.1}$$

$$s.t.\ \dot{x} = Ax + Bu,\ x(t) = x,\ s \in [t, T],\ x \in R^n,\ p(u) \le 1. \tag{2.2}$$

Define the value function [9] with respect to the primal problem $(\mathcal{P})$

$$V(t, x) = \inf J(t, x, u). \tag{2.3}$$

By classical principle of optimality, we know that [4,6,9], the value function $V(t, x)$(defined in (2.3)) is just the viscosity solution to the Hamilton–Jacobi–Bellman equation

$$-v_t(t, x) = \inf_{p(u)\le 1} \{v_x^T(t, x)[Ax + Bu] + F(x)\}, \tag{2.4}$$

with the boundary condition $v(T, x) = Q(x)$. Indeed, by classical theory of viscosity solution [4,6] to HJB equation, noting that the state set of the linear system is bounded when the control taking value in a compact set, one can show that the value function $V(t, x)$ is a unique viscosity solution to the Eq. (2.4). For computing a viscosity solution to the HJB equation (2.4), we present a Cauchy initial value problem for given small positive real number ϵ:

$$\epsilon \Delta_x v(t, x) = v_t(t, x) + \inf_{p(u)\le 1} \{v_x^T(t, x)[Ax + Bu] + F(x)\}, \tag{2.5}$$

with the boundary condition $v(T, x) = Q(x)$. When $0 < \epsilon \ll 1$, the Cauchy initial value problem (2.5) is called a viscosity approximation to the Hamilton–Jacobi–Bellman equation (2.4). It has been proved for the convergence of (2.5)–(2.4) in the classical theory of viscosity solution to HJB equation [4,6]. In the rest of the paper we study how to compute the viscosity approximation to the Hamilton–Jacobi–Bellman equation (2.5).

Checking the right side of the Eq. (2.5), we find that the inf process works on the control variable u and is not relevant to the state variable x and the time variable t. For given x and t we consider to deal with the optimization(for obtaining a global minimizer):

$$\min_{p(u)\le 1} [v_x^T(t, x)Bu]. \tag{2.6}$$

Therefore we turn to consider the following optimization with respect to a given parameter vector $\lambda \in R^m$:

$$\min_{p(u)\le 1} [\lambda^T u]. \tag{2.7}$$

Remark 2.1. Note that, when $\lambda = 0$, the optimization in (2.7) is trivial and the corresponding minimizer of the optimization will be granted to be the critical point of $p(u)$ [3]. On the other hand, when $\lambda \neq 0$, the minimizer can not be in the interior of U, otherwise at the minimizer we have $\nabla_u[\lambda^T u] = \lambda = 0$ to meet a contradiction. Consequently when $\lambda \neq 0$, if u^* is a minimizer, then $p(u^*) = 1$ (noting that $p(u^*) < 1$ implies that u^* is an interior point of U). Next we claim that $\nabla p(u^*) \neq 0$. In fact, if $\nabla p(u^*) = 0$, then u^* is the unique minimizer of $p(u)$ over R^m noting that $\nabla^2 p(u) > 0$ over R^m. It implies that $U = \{u : p(u) \le 1\}$ consists of only one point. It is a trivial case. Being a nonzero vector $\nabla p(u^*)$, it is linear independent. By classical optimization theory, the minimizer of $\lambda^T u$ over U is a KKT point [5]. Thus there exists a positive number ρ^* such that $\lambda + \rho^* \nabla p(u^*) = 0$.

3. A minimizer flow

In this section, for given $\lambda \in R^m$ we seek a minimizer u of the optimization problem (2.7) to create a function $u = h(\lambda)$ which is bounded and measurable. In what follows we present a differential flow to define $h(\lambda)$.

Lemma 3.1. *For given $\lambda \in R^m$ and $\rho > 0$, there exists a unique $u_{\lambda,\rho} \in R^m$ such that $\lambda + \rho \nabla p(u_{\lambda,\rho}) = 0$.*

Proof. With the assumption (1.3), by Lemma 2.1 in [10], there exists a unique minimizer of the function $\frac{\lambda}{\rho} u + p(u)$ over R^m noting that $\rho > 0$ here. Consequently, there exists a unique point $u_{\lambda,\rho} \in R^m$ such that $\lambda + \rho \nabla p(u_{\lambda,\rho}) = 0$ noting the assumption for the optimal control problem $(\mathcal{P})$: $\nabla^2 p(u) > 0$, $\forall u \in R^m$. The lemma has been proved.

Associated with the optimization problem (2.7), a differential flow to solve the optimization problem (2.7) is presented as follows. Define a Lagrange function

$$L(\rho, u) = \lambda^T u + \rho(p(u) - 1).$$

We have

$$\nabla_u L(\rho, u) = \lambda + \rho \nabla p(u). \tag{3.1}$$

By Lemma 3.1, for $\rho = 1$, we can solve the equation

$$\nabla_u L(1, u) = \lambda + \nabla p(u) = 0, \tag{3.2}$$

to obtain a unique solution $u = u_\lambda$. Since $\nabla^2 p(u) > 0$, $\forall u \in R^m$, u_λ is continuously dependent on λ. Further we conclude the following result. □

Theorem 3.1. *The differential flow $\hat{u}(\rho)$, which is well defined on $(0, +\infty)$ by the ordinary differential equation*

$$\frac{du}{d\rho} + [\rho \nabla^2 p(u)]^{-1} \nabla p(u) = 0,\ u(1) = u_\lambda, \tag{3.3}$$

satisfies

$$\lambda + \rho \nabla p(\hat{u}(\rho)) \equiv 0. \tag{3.4}$$

Proof. Since $\nabla^2 p(u) > 0$, $\forall u \in R^m$, for $\rho \in (0, +\infty)$ there exists a unique solution to the following ordinary differential equation (3.3) [2]. Then we have for $\rho \in (0, +\infty)$