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Necessary and Sufficient Conditions for Perfect Command Following and Disturbance Rejection in Fractional Order Systems

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Abstract: The aim of this paper is to present a modified explanation of the classic internal model principle for certain class of finite-dimensional, time-invariant, deterministic fractional-order systems commonly known as fractional systems of commensurate order. The necessary and sufficient conditions for perfect command tracking and disturbance rejection are provided. The difficulty of applying the classic internal model principle to fractional-order systems is due to the difference between integer-order and fractional-order systems from the zero-pole cancellation point of view. The notion of zero-pole cancellation is discussed for the systems under consideration in a well posed mathematical framework. It is also shown that fractional elements can be used for command tracking and disturbance rejection purposes which provides more flexibility for controller design applications. Two illustrative examples confirm the applicability of the proposed theorems.

Keywords: Fractional systems; Controller constraints and structure; Analytic design; Internal model principle.

1. INTRODUCTION

The idea of internal model principle was first introduced in the work of Francis and Wonham (1976) which dealt with the regulator problem for linear, time-invariant, finite-dimensional systems with deterministic disturbance and reference signals. The main result of that work, for the closed-loop system shown in Fig. 1, is that the controller $C(s)$ must incorporate in the feedback path a suitable model of the dynamic structure of the disturbance and reference signal in order to achieve perfect asymptotic disturbance rejection and command tracking. That is why an integrator must be provided in the forward path of a given stable closed-loop system for tracking the step input without steady-state error.

In recent years there has been an increasing attention to fractional-order systems. These systems are of interest for both modelling and controller-design purposes. In the field of continuous-time modelling, fractional derivatives have proved to be useful in linear viscoelasticity, acoustics, rheology, polymeric chemistry, biophysics, etc (Oldham and Spanier, 1974; Hilfer, 2000). In general, fractional-order systems are useful to model various stable physical phenomena (commonly diffusive type systems) with anomalous decay, say those that are not of an exponential type. For example, Miller and Ross (1993) introduced a real-world system with impulse response

$$h(t) = \frac{\sqrt{2g\pi}}{\Gamma(3/2)} t_+^{1/2}, \tag{1}$$

which corresponds to the transfer function

$$H(s) = \frac{\sqrt{2g\pi}}{s^{3/2}}. \tag{2}$$

As an example of using fractional derivatives for modelling, Beyer and Kempfle (1995) studied the generalized damping equation

$$(D^2 + aD^q + b)x(t) = f(t), \quad q \in (0, 2) \tag{3}$$

and discussed the advantages of fractional modelling. The transfer function of the above system is easily found to be

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + as^q + b}. \tag{4}$$

In the field of linear viscoelasticity, Glöckle et al. (1991) used fractional calculus to generalize the Zener model. They proposed the fractional (integral) equation of unknown orders β and μ:

$$\frac{1}{\tau_0^\beta}\, {}_0D_t^{-\beta}\sigma(t) + \sigma(t) - \sigma_0 = \frac{G_e}{\tau_0^\beta}\, {}_0D_t^{-\mu}\varepsilon(t) + G_0[\varepsilon(t) - \varepsilon_0], \tag{5}$$

where σ and ε are stress and strain, respectively, and τ_0, G_m, η_m, and G_e are real physical constants. Equation (5) corresponds to the transfer function

$$H(s) = \frac{\widetilde{\sigma}(s)}{\widetilde{\varepsilon}(s)} = \frac{G_0 + G_e(s\tau_0)^{-\mu}}{1 + (s\tau_0)^{-\beta}}, \tag{6}$$

where the initial values are chosen such that $\sigma_0 = G_0\varepsilon_0$. The transfer functions (2), (4), and (6) represent practical systems with non-integer powers of the Laplace variable.

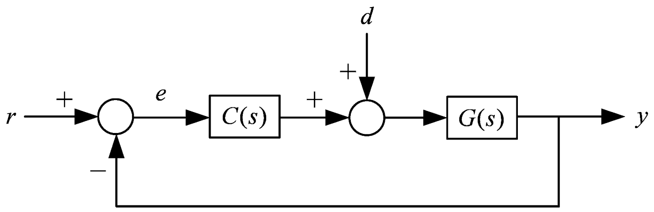

Fig. 1. The standard closed-loop system

An interesting study of fractional differential systems appeared in (Viano et al., 1994) using a stochastic framework. The idea of fractional powers is also used for identification purposes in order to reach more accurate models. Tsao et al. (1989) and Poinot and Trigeassou (2004), clarify the identification method when the members of model set are of fractional order. Two applications of such identifications can be found in (Vinagre et al., 1998) and (Chauchois et al., 2003). Fractional differential systems are also used in control field. Podlubny (1999) and Valério and Costa (2006) discussed methods of designing $PI^{\lambda}D^{\mu}$ controllers, Raynaud and ZergaInoh (2000) studied fractional-order lead-lag compensators and Oustaloup et al. (1995, 1996) introduced the so-called CRONE controllers.

Systems of *commensurate order* of derivatives are the systems that have been described by fractional differential equations of commensurate order. Such systems lend themselves well to some algebraic tools (Miller and Ross, 1993; Beyer and Kempfle, 1995). For instance, $H(s)$ as defined in (2) is a transfer function for a system of commensurate order. More examples of practical fractional differential systems of commensurate order can be found in (Beyer and Kempfle, 1995; Vinagre et al., 1998; Chauchois et al., 2003). The inverse Laplace transform of such systems involve special functions (for definition and notations see Miller and Ross, 1993).

It was shown in (Francis and Wonham, 1976) that the purpose of the internal model is to supply closed-loop transmission zeros which cancel the unstable poles of the disturbance and reference signals. But unfortunately the notion of zero-pole cancellation in fractional case (e.g., in dealing with transfer functions like (2), (4), or (6)) is much more different from the integer case. Note that unlike the integer case, if $A(s)$ and $B(s)$ are two fractional-order polynomials (see Definition 1) with the same zeros, then in general we cannot conclude that $A(s)/B(s)$ is equal to a constant value, i.e. a zero does not necessarily cancel the same pole. For example, consider $A(s) = s^{1/2} - 1$ and $B(s) = s^{1/3} - 1$. Both A and B have only one zero at $s = 1$ (see Proposition 3), but

$$\begin{aligned}\frac{A(s)}{B(s)} &= \frac{s^{1/2} - 1}{s^{1/3} - 1}\\ &= \frac{(s^{1/6} - 1)(s^{1/3} + s^{1/6} + 1)}{(s^{1/6} - 1)(s^{1/6} + 1)}\\ &= \frac{s^{1/3} + s^{1/6} + 1}{s^{1/6} + 1} \neq \text{constant.}\end{aligned}$$

This example shows the need for a modified explanation of the existing internal model principle which is discussed in this paper. The aim of this brief is not to propose a controller synthesis algorithm but only to provide the necessary and sufficient conditions needed for perfect command tracking and disturbance rejection in fractional case.

The rest of this paper is divided to four sections. Problem preliminaries are presented in Section 2. Theorems 8 and 9 are the main results of this paper which provide the necessary and sufficient conditions for perfect command tracking and disturbance rejection for fractional systems under consideration. These two theorems are studied in Section 3. Two illustrative examples are presented in Section 4 and finally, Section 5 contains the conclusion.

2. PRELIMINARIES

2.1 Problem Prerequisites

Before introducing the main problem, some definitions and notations are provided. For simplicity, the "fractional system of commensurate order" will be addressed by "fractional system" in the rest of this paper.

Definition 1. The function

$$Q(s) = a_1 s^{q_1} + a_2 s^{q_2} + \ldots + a_n s^{q_n}, \tag{7}$$

is a fractional-order polynomial if and only if $q_i \in \mathbb{Q}^+ \cup \{0\}, a_i \in \mathbb{R}, i = 1 \ldots n$, where $\mathbb{Q}^+$ and $\mathbb{R}$ stand for the sets of positive rational numbers and real numbers, respectively.

Definition 2. Consider the fractional-order polynomial

$$Q(s) = a_1 s^{\frac{\alpha_1}{\beta_1}} + a_2 s^{\frac{\alpha_2}{\beta_2}} + \ldots + a_n s^{\frac{\alpha_n}{\beta_n}}, \tag{8}$$

where

$$a_i \in \mathbb{R}, \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad \beta_i \in \mathbb{N},$$

and α_i, β_i are relatively prime for $i = 1, \ldots, n$ and $\mathbb{N}$ is the set of natural numbers. (If for some i, $\alpha_i = 0$ then by definition $\beta_i = 1$.) Let λ be the least common multiple (lcm) of $\beta_1, \beta_2, \ldots, \beta_n$ denoted as $\lambda = \text{lcm}\{\beta_1, \beta_2, \ldots, \beta_n\}$. Then

$$Q(s) = a_1 s^{\frac{\lambda_1}{\lambda}} + a_2 s^{\frac{\lambda_2}{\lambda}} + \ldots + a_n s^{\frac{\lambda_n}{\lambda}} \tag{9}$$

$$= a_1 (s^{\frac{1}{\lambda}})^{\lambda_1} + a_2 (s^{\frac{1}{\lambda}})^{\lambda_2} + \ldots + a_n (s^{\frac{1}{\lambda}})^{\lambda_n}. \tag{10}$$

Now the fractional degree (fdeg) of $Q(s)$ is defined as $\text{fdeg}\{Q(s)\} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

The domain of definition for (10) is a Riemann surface with finite number of Riemann sheets (λ sheets here) where origin is a branch point (of order $\lambda - 1$) and the branch-cut is assumed at $\mathbb{R}^-$ (LePage, 1961). Note that the fractional-order polynomial and the fractional degree as defined above reduce to the conventional concepts of polynomial and the degree of a polynomial when $\lambda = 1$. The following proposition gives the roots number for a fractional algebraic equation.

Proposition 3. Let $Q(s)$ be a fractional-order polynomial with $\text{fdeg}\{Q(s)\} = n$. Then the equation $Q(s) = 0$ has exactly n roots on the Riemann surface.

Proof. Consider

$$Q(s) = a_1 (s^{\frac{1}{v}})^n + a_2 (s^{\frac{1}{v}})^{n-1} + \ldots + a_n (s^{\frac{1}{v}})^1 + a_{n+1}, \tag{11}$$

for an appropriate $v \in \mathbb{N}$. Assuming $w := s^{\frac{1}{v}}$, we have

$$\widetilde{Q}(w) = a_1 w^n + a_2 w^{n-1} + \ldots + a_n w + a_{n+1}. \tag{12}$$

The fundamental theorem of algebra gives n roots for $\widetilde{Q}(w) = 0$, say $w_1, w_2, \ldots, w_n$. Consequently, $Q(s) = 0$ has n roots at $s_1 = w_1^v, s_2 = w_2^v, \ldots, s_n = w_n^v$.