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# Deterministic and Stochastic Newton-based extremum seeking for higher derivatives of unknown maps with delays☆

Damir Rušiti[a,*], Tiago Roux Oliveira[b], Greg Mills[c], Miroslav Krstić[c]

[a] Department of Information-oriented Control (ITR), Technical University of Munich (TUM), Munich, Germany
[b] Department of Electronics and Telecommunication Engineering, State University of Rio de Janeiro (UERJ), Rio de Janeiro, Brazil
[c] Department of Mechanical and Aerospace Engineering, University of California - San Diego (UCSD), La Jolla, CA, USA

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ABSTRACT

We present a Newton-based extremum seeking algorithm for maximizing higher derivatives of unknown maps in the presence of time delays using deterministic perturbations. Different from previous works about extremum seeking for higher derivatives, arbitrarily long input-output delays are allowed. We incorporate a predictor feedback with a perturbation-based estimate for the Hessian's inverse using a differential Riccati equation. As a bonus, the convergence rate of the real-time optimizer can be made user-assignable, rather than being dependent on the unknown Hessian of the higher-derivative map. Averaging method for arbitrary shaped derivatives under delays is presented. Exponential stability and convergence to a small neighbourhood of the unknown extremum point are achieved for locally quadratic derivatives by using a backstepping transformation and averaging theory in infinite dimensions. Furthermore, we give a brief introduction into stochastic Newton-based Extremum Seeking for constant output delays, where we show the differences and similarities with respect to the deterministic case. We also present illustrative numerical examples in order to highlight the effectiveness of the proposed predictor-based extremum seeking for time-delay compensation applying both deterministic and stochastic perturbations.



## 1. Introduction

Extremum Seeking (ES) is a non-model based and real-time optimization technique for nonlinear equilibrium maps that impose local optimum, either minimum or maximum. In recent years, there have been lots of publications on ES in theory [5,7,9,13,21,22,30,31] as well as applications [6,26,27,29,33]. In [8], Newton-based ES (free of delays) was deeply studied. A highlight of these works is the approach used to estimate the Hessian's inverse of the nonlinear map, which is generated by means of a Riccati filter. This is applied to remove the dependence of the algorithm's convergence rate on the second derivative (Hessian), making it user-assignable. The results mentioned above deal only with extremum seeking for the map itself.

However, there are applications where an extremum of the map's higher derivative is sought. In [32] the authors present a refrigeration system where a suitable operating point is located at the maximum negative slope that is subject to change. This point of zero curvature corresponds to a minimum of the first derivative of the input-output map. Hence, being able to track the minimum of the first derivative in real-time would allow the system to operate almost the whole time at the most suitable operating condition.

A Newton-based ES generalization was presented in [19] to maximize arbitrary higher derivatives of an unknown map. Using periodic perturbations, estimation of the gradient and the Hessian of map's $n^{\text{th}}$ derivative were obtained as well as the local stability proof for the closed-loop system. However, reference [19] does not cope with maps under delays. Time delays are some of the most common phenomena that arise in engineering practice and need to be handled carefully since even small delays may result in a degradation of the system's behaviour or even lead to instability. The first publications that deal with Newton-based ES in the presence of constant and known time delays are [23–25], where only the extremum of a map was sought and an not the maximization-minimizati-on of its derivatives.

In this paper, we extend the applicability and usage of the predictor-based controller with an averaging-based estimate of the Hessian's inverse proposed in [23–25] to maximize higher derivatives of a static map despite the presence of time delays [28]. Our

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* Corresponding author.
*E-mail addresses:* damir.rusiti@tum.de (D. Rušiti), tiagoroux@uerj.br (T.R. Oliveira), g1mills@ucsd.edu (G. Mills), krstic@ucsd.edu (M. Krstić).

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generalization for the Newton optimization uses the Hessian estimate of the map's higher derivative for the purpose of implementing a predictor that compensates the delay and makes the convergence rate independent of the unknown parameters of the nonlinear map. The convergence properties of the ES algorithm for maximizing arbitrary shaped derivatives by using only measurements of the map are outlined first in the spirit of finite spectrum assignment. After that, we rigorously prove the stability for locally quadratic derivatives via backstepping transformation [14] and averaging theory in infinite dimensions [10,16], considering the whole system which is infinite dimensional due to delays.

As an additional contribution, we give a brief introduction into the stochastic generalization of the Newton-based ES algorithm with constant output delays, stating the main differences with respect to the deterministic case in terms of design and analysis. There are clear benefits in investigating the use of stochastic perturbations over the deterministic ES architecture, as discussed in [18]. For instance, limitations of the deterministic ES scheme include the fact that learning using a periodic excitation signal is rather simple-minded and rare in probing-based learning and optimization approaches [17], which may lead to slower converge rates. In addition, ES algorithms inspired by bio-mimicry [20] and others sensitive to deterministic perturbation signals [3] suggest other perturbation techniques using random motion rather then periodic ones.

At the last, numerical simulations show the applicability of the proposed algorithms in online maximization-minimization problems as well as comparison results for both deterministic and stochastic perturbations.

– *Notation and terminology* – The 2– norm of a finite-dimensional (ODE) state vector $X(t)$ is denoted by single bars, $|X(t)|$. In contrast, norms of functions (of $x$) are denoted by double bars. By default, $\|\cdot\|$ denotes the spatial $L_2[0, D]$ norm, *i.e.*, $\|\cdot\| = \|\cdot\|_{L_2[0,D]}$. Since the PDE state variable $u(x, t)$ is a function of two arguments, we should emphasize that taking a norm in one of the variables makes the norm a function of the other variable. For example, the $L_2[0, D]$ norm of $u(x, t)$ in $x \in [0, D]$ is $\|u(t)\| = (\int_0^D u^2(x,t)dx)^{1/2}$.

The partial derivatives of $u(x, t)$ are denoted by $u_t(x, t)$ and $u_x(x, t)$ or, occasionally, by $\partial_t u_{\text{av}}(x, t)$ and $\partial_x u_{\text{av}}(x, t)$ to refer the operator for its average signal $u_{\text{av}}(x, t)$. Consider a generic nonlinear system $\dot{x} = f(t, x, \epsilon)$, where $x \in \mathbb{R}^n$, $f(t, x, \epsilon)$ is periodic in $t$ with period $\Pi$, i.e., $f(t+\Pi, x, \epsilon) = f(t, x, \epsilon)$. Hence, for $\epsilon > 0$ sufficiently small, we can obtain its average model given by $\dot{x}_{\text{av}} = f_{\text{av}}(x_{\text{av}})$, with $f_{\text{av}}(x_{\text{av}}) = 1/\Pi \int_0^\Pi f(\tau, x_{\text{av}}, 0)d\tau$, where $x_{\text{av}}(t)$ denotes the average version of the state $x(t)$ [12].

As defined in [12], a vector function $f(x, \epsilon) \in \mathbb{R}^n$ is said to be of order $\mathcal{O}(\epsilon)$ over an interval $[t_1, t_2]$ if there exist positive constants $k$ and $\epsilon^*$ such that $|f(t, \epsilon)| \le k\epsilon$, $\forall \epsilon \in [0, \epsilon^*]$ and $\forall t \in [t_1, t_2]$. In this manuscript we will be referring to $\mathcal{O}(\epsilon)$ being an order of magnitude relation, which is valid for $\epsilon$ sufficiently small. Moreover, we define any arbitrary initial time as $t_0 \ge 0$.

## 2. Newton-based extremum seeking of higher derivatives under delays

Scalar ES considers applications in which one wants to maximize (or minimize) the output $y \in \mathbb{R}$ of an *unknown* nonlinear static map $h(\theta)$ by varying the input $\theta \in \mathbb{R}$ in *real time*. But like in many technical applications we have to consider that the output may be time-delayed [2], and hence, we additionally assume that there is a constant and known delay $D \ge 0$ such that the output is expressed by

$$y(t) = h(\theta(t-D)). \tag{1}$$

In this paper, we assume that our system is output-delayed. Since any input delay can be moved to the output of a static map, the results from this paper can be directly extended to the input-delay case. Also the case when input delays $D_{\text{in}}$ and output $D_{\text{out}}$ delays occur simultaneously can be handled by assuming that the total delay is $D = D_{\text{in}} + D_{\text{out}}$, with $D_{\text{in}}, D_{\text{out}} \ge 0$. Furthermore, we only consider measurements without noise and/or disturbances that is not an objective of this paper and should be handled separately.

Without loss of generality, let us consider the maximization of $n^{\text{th}}$ derivative of the output in the presence of time delay using Newton-based ES, where the maximizing value of $\theta$ is denoted by $\theta^*$. We state our optimization problem as follows:

$$\max_{\theta \in \mathbb{R}} h^{(n)}(\theta(t-D)), \tag{2}$$

with nonlinear map $h(\cdot)$ satisfying the next assumption.

**Assumption 1.** Let $h^{(n)}(\cdot)$ be the $n^{th}$ derivative of a smooth function $h(\cdot)$: $\mathbb{R} \to \mathbb{R}$. Now let us define

$$\Theta_{max} = \{\theta | h^{(n+1)}(\theta) = 0, \quad h^{(n+2)}(\theta) < 0\} \tag{3}$$

to be a collection of maxima where $h^{(n)}$ is locally concave. Now assume that $\exists \theta^* \in \Theta_{max}$ and $\Theta_{max} \neq \emptyset$.

In Fig. 1, we illustrate the proposed scalar version of the Newton-based ES for maximization of higher derivatives based on predictor feedback for delay compensation. The design parameters are $k$, $k_R$, $a$, $\omega$, $c > 0$ as presented in Fig. 1. According to [19], we switch from maximization to minimization problem by setting $\text{sgn}(\gamma_0) = \text{sgn}(h^{(n+2)}(\theta^*))$ with $\gamma_0$ as initial value of $\gamma$.

### 2.1. System and signals

Let $\hat{\theta}$ be the estimate of the maximizer and

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^* \tag{4}$$

be the estimation error. From the block diagram in Fig. 1, the error dynamics can be written as

$$\dot{\tilde{\theta}}(t) = \dot{\hat{\theta}}(t) = U(t). \tag{5}$$

Moreover, we have

$$\dot{\gamma} = k_R \gamma (1 - \gamma \widehat{h^{(n+2)}}), \tag{6}$$

where (6) is a differential Riccati equation. Eq. (6) will be used to generate an estimate of the Hessian's inverse [8] according to the following error transformation

$$\tilde{\gamma} = \gamma - \underbrace{\frac{1}{h^{(n+2)}(\theta^*)}}_{H^{-1}}. \tag{7}$$

Rearranging the equations given in [19] for the block diagram in Fig. 1 including delays, we can write:

$$\theta = \hat{\theta} + a\sin(\omega t), \tag{8}$$

$$\Upsilon_j = C_j \sin\left(j\omega t + \frac{\pi}{4}\left(1 + (-1)^j\right)\right), \tag{9}$$

$$C_j = \frac{2^j j!}{a^j}(-1)^F, \tag{10}$$

$$F = \frac{j - \left|\sin\left(\frac{j\pi}{2}\right)\right|}{2}. \tag{11}$$

We have defined the additive dither signal as

$$S(t) = a\sin(\omega t) \tag{12}$$

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