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Local null controllability of the control-affine nonlinear systems with time-varying disturbances

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ABSTRACT

The problem of local null controllability for the control-affine nonlinear systems $\dot{x}(t) = f(x(t)) + Bu(t) + w(t)$, $t \in [0, T]$ is considered in this paper. The principal requirements on the system are that the LTI pair $((\partial f/\partial x)(0), B)$ is controllable and the disturbance is limited by the constraint $|f(0) + w(t)| \leq M_d\left(1 - \frac{t}{T}\right)^{\eta}$, $M_d \geq 0$ and $\eta > 0$. These properties together with one technical assumption yield an answer to the problem of deciding when the null controllable region has a nonempty interior. The obtained criterion is built on the purely algebraic and/or differential manipulations with vector field f, input matrix B and a bound on the disturbance $w(t)$. To prove the main result we have derived a new Gronwall-type inequality allowing the fine estimates of the closed-loop solutions. The theory is illustrated and the efficacy of proposed controller is demonstrated by the example where the null controllable region is explicitly calculated. Finally, we established the sufficient conditions to be the system under consideration with $w(t) \equiv 0$ globally null controllable.

1. Introduction

In this paper we will concern ourselves with the problem of null controllability of the nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) + Bu(t) + w(t), \ \ t \in [0, T], \ \ x(0) = x_0 \in \mathbb{R}^n, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^n$ represents the total disturbance (unmodeled system dynamics, uncertainty, overall external disturbances that affect the system, etc.) which is potentially unknown but with known magnitude constraint $|f(0) + w(t)| \leq M_d\left(1 - \frac{t}{T}\right)^{\eta}$ for some T, $M_d \geq 0$ and $\eta > 0$, specified below in the proof of Lemma 3.1. The function f is C^2 on $\mathbb{R}^n$ and B is an $n \times m$ constant matrix. We establish the sufficient conditions for the existence and the method for determining the approximations of null controllable region $\mathcal{X}_0$ in the sense of the definitions below. Henceforth, we use the following notations: The $n \times n$ matrix $(\partial f/\partial x)(0)$ is a Jacobian matrix of the vector field $f(x)$ evaluated at $x = 0$ and an upper dot denotes the derivative relative to time t. The superscript $'\top'$ is used to indicate the transpose operator. We denote by $|\cdot|$ the Euclidean norm and by $||\cdot||$ a matrix norm induced by the Euclidean norm of vectors, $||A|| = \max_{|x|=1} |Ax|$. It is well-known, see e.g. [19], that this norm is equivalent to the spectral norm for matrices, $||A|| = \sqrt{\lambda_{max}(A^\top A)}$. The real part of a complex number z is denoted by $\Re(z)$. Further, we shall always assume that the domain of existence of trajectories for the control system (1) is at least the interval $[0, T]$ for every x_0, every continuous input $u(t)$ and the continuous disturbance $w(t)$ satisfying the constraint above.

The properties of the systems related to controllability have been analyzed by many researchers for different meaning, among them the concept of null controllability. We will use the following definition from [7] that we modified for our purposes.

Definition 1.1. The system (1) is said to be *locally null controllable* if there exists an open neighborhood $\mathcal{X}_0$ of the origin in $\mathbb{R}^n$ and a finite time $T > 0$ such that, to each $x_0 \in \mathcal{X}_0$, there corresponds a continuous function $u : [0, T] \to \mathbb{R}^m$ such that the solution $x(t)$ of (1) determined by this $u = u(t)$ and $x(0) = x_0$ satisfies $x(T) = 0$.

In general, the concept of controllability is defined as an open-loop control, but in many situations a state feedback control is preferable. The definition is as follows.

Definition 1.2. The system (1) is said to be *locally null controllable by a state feedback controller* if there exists an open neighborhood

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$\mathcal{X}_0$ of the origin in $\mathbb{R}^n$ and a finite time $T > 0$ such that, to each $x_0 \in \mathcal{X}_0$, there corresponds a continuous control law $v(x, t)$ such that the solution $x(t)$ of (1) determined by $u = v(x, t)$ and $x(0) = x_0$ satisfies $x(T) = 0$.

The majority of results for controllability have been established for linear time-invariant (LTI) systems $\dot{x} = Ax + Bu$, where Kalman [10,11] has shown that a necessary and sufficient condition for global ($\mathcal{X}_0 = \mathbb{R}^n$) null controllability is

$$\operatorname{rank}\left(B, AB, \ldots, A^{n-1}B\right) = n,$$

or equivalently, the controllability Gramian matrix $W_c(t_1)$,

$$W_c(t_1) \triangleq \int_0^{t_1} P(t)BB^{\mathrm{T}}P^{\mathrm{T}}(t)dt,$$

is invertible for any $t_1 > 0$. Here $P(t) = e^{At}$ is a fundamental matrix of homogeneous system $\dot{x} = Ax$.

The situation is more delicate for linear time-variant (LTV) systems $\dot{x} = A(t)x + B(t)u$. Although there is a well-known Gramian matrix-based criterion for the (global) null controllability of such systems, but for a general LTV system there is no analytical expression that expresses $P(t)$ as a function of $A(t)$. Nevertheless, in [2] (Theorem 2. 1) has been proved that small perturbations $V(t)$ and $B(t)$ of constant matrices A and B, respectively, preserves the (global) null controllability.

In terms of nonlinear system controllability, one of the most important results in this field was derived by Lee and Markus [13,17]. The result states that if a linearized system $\dot{x} = Ax + Bu$ at an equilibrium point (0, 0) $[A = (\partial f/\partial x)(0, 0)$ and $B = (\partial f/\partial u)(0, 0)]$ is controllable, then there exists a local controllable area of the original nonlinear system $\dot{x} = f(x, u)$ around this equilibrium point.

Later, it turned out that that fact is also true in the case when the linearized system is time-varying [6, p. 127] and this result gives us a good reason to locally use linearized system instead of the original nonlinear system. In particular, this applies when **(i)** the system is linearized around equilibrium point, in which case the matrices A, B are constants, and the controllability of LTI systems is easy to verify, or **(ii)** the results of global controllability do not hold or are not easy to be obtained. The drawback of this approach is that the fundamental theorems do not refer on the region where we can use the linearized systems instead of the original nonlinear systems. Some of the few papers concerning with this topic are [12,16], or [8] for LTI systems with a constrained input.

Completely different principles and techniques than those based on the linearization around the trajectory of control system are behind the geometric control theory. This theory, for the time-invariant systems $\dot{x} = f(x, u)$, establishes a connection between the Lie algebras of vector fields and the sets of points reachable by following flows of vector fields. For your reference, see e.g. the pioneering works [3–5,14,15] and [21] or the now classical monographs [9,18,20]. The standard assumption that is made throughout these works is that f is an analytic function of the variable x. This analyticity assumption cannot be relaxed without destroying the theory as was carefully analyzed and emphasized in [22].

Unfortunately, none of this theories is not applicable to the systems considered in the present paper in general, and to the best of our knowledge, there has been probably very limited (if any) research on null controllability of nonlinear systems with time-varying disturbances - and so this topic does not seem to have been well studied until now. Moreover, the technique of the proof of the key Lemma 3.1 (Section 3) allows us to explicitly estimate the null controllable region (Section 4, Example 4.1).

In the following section, we formulate the important technical result, the new Gronwall-type inequality, used in the proof of Lemma 3.1 providing a quantitative estimate of the solutions to system (1) on the interval [0, T].

2. Technical result: Gronwall-type inequality

Lemma 2.1. *(Compare with [1, p. 35]) If $u_1(\tau), v_1(\tau), w_1(\tau) \geq 0$ are the continuous functions for all $\tau \geq 0$, if c_1 is a positive constant, and if*

$$u_1(\tau) \leq c_1 + \int_0^\tau (u_1(s)v_1(s) + w_1(s))ds \tag{2}$$

then

$$u_1(\tau) \leq e^{\int_0^\tau v_1(s)ds}\left[c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right)\right], \ \forall \tau \geq 0, \ \forall \rho > 0. \tag{3}$$

Proof. From (2) and the inequality $e^z \geq z + 1$ for $z = \int_0^\tau \frac{w_1(s)}{\rho}ds \geq 0$ we have for all $\rho > 0$

$$u_1(\tau) \leq c_1 + \rho e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - \rho + \int_0^\tau u_1(s)v_1(s)ds \tag{4}$$

which implies

$$\frac{u_1(\tau)}{c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right) + \int_0^\tau u_1(s)v_1(s)ds} \leq 1.$$

Multiplying this with $v_1(\tau) \geq 0$ we obtain

$$\frac{u_1(\tau)v_1(\tau)}{c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right) + \int_0^\tau u_1(s)v_1(s)ds} \leq v_1(\tau)$$

and

$$\frac{u_1(\tau)v_1(\tau) + w_1(\tau)e^{\int_0^\tau \frac{w_1(s)}{\rho}ds}}{c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right) + \int_0^\tau u_1(s)v_1(s)ds} \leq v_1(\tau) + \frac{w_1(\tau)e^{\int_0^\tau \frac{w_1(s)}{\rho}ds}}{c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right)}.$$

Integrating both sides between 0 and τ we have

$$\ln\left[c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right) + \int_0^\tau u_1(s)v_1(s)ds\right] - \ln c_1 \leq \int_0^\tau v_1(s)ds + \ln\left[c_1 + \rho\left(e^{\int_0^\tau \frac{w_1(s)}{\rho}ds} - 1\right)\right] - \ln c_1.$$

Converting this to exponential form and taking into considerations (4), the inequality becomes (3). □

3. The auxiliary lemmas and main result

Before stating the main result of the present paper we prove the following important lemma.

Lemma 3.1. *Let us consider the control system (1). Assume that*

(H1) *the LTI pair (A, B), $A = (\partial f/\partial x)(0)$, is controllable;*
(H2) *there exist an $m \times n$ constant matrix K and the constants $k_1 \geq 1$, $\Gamma_0 > 0$, $\varepsilon > 0$, $\tilde{\lambda}(K) > 0$, $\sigma > 0$, $\delta \leq 0$, $M_d \geq 0$ and the time $T > 0$ such that*

$$k_1(\Gamma_0 + T||A||) + \sigma \leq \tilde{\lambda} < k_1T||A|| + \sigma - \delta \ (\text{for } M_d > 0) \tag{5}$$

or

$$k_1(\Gamma_0 + T||A||) + \sigma \leq \tilde{\lambda} \ (\text{for } M_d = 0), \tag{5'}$$