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Stabilization of photon-number states via single-photon corrections: A convergence analysis under imperfect measurements and feedback delays[☆]

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ABSTRACT

This paper presents a mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of the Laboratoire Kastler Brossel. Here, we consider an infinite-dimensional Markov model corresponding to a realistic experimental set-up where imperfect measurements and feedback delays are taken into account. In this realistic context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number subspace. Closed-loop simulations illustrate the performance of the feedback law.

1. Introduction

A photon-number states (Fock states) feedback stabilization scheme via single-photon corrections was described and experimentally tested in [11] by the cavity quantum electrodynamics group of the Laboratoire Kastler Brossel (LKB).[1] Such control problem is relevant for quantum information applications [4,7]. The quantum state ρ corresponds to the density operator of a microwave field stored inside a super-conducting cavity and described as a quantum harmonic oscillator. At each sample step $k \in \mathbb{N}$, a probe atom is launched inside the cavity. The measurement outcome y_k detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the ground energy state $|g\rangle$, or the excited energy state $|e\rangle$. Consequently, the measurement outcomes correspond to a discrete-valued output y_k with only 2 distinct possibilities: g or e. Similarly, the control inputs u_k are also discrete-valued with 3 distinct possibilities: $-1, 0, +1$. The open-loop value $u_k = 0$ corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition (QND) measurement of Fock states [2]. The two other values $u_k = \pm 1$ correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Up to now, despite the successful experimental implementation achieved in [11], there is no mathematical convergence proof: this is due to the fact that the control values are discrete (only 3 different values) and the system state is continuous (density operator). This paper establishes a first result assuming that the initial condition is known and has a support of finite photon-number, which may open the way to more complete results where it is unknown and its support involves an infinite number of photons and specific Banach spaces of trace-class operators. The main interest for future applications relies on the fact that the open-loop measurement process is QND: quantum non-demolition measurements are widely used in feedback schemes underlying error corrections (see e.g. [6]).

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[1] Serge Haroche, who was awarded the Nobel Prize in Physics 2012, is a member of the cavity quantum electrodynamics group of LKB and is one of the coauthors of [11].

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This paper shows that, by adding an arbitrarily small term to the Lyapunov function used in [11], one ensures almost sure global stabilization of any goal Fock state for the closed-loop quantum system (see Theorem 2 in Section 3.2). This is achieved by relying on an infinite-dimensional Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [11]. Such model takes into account the back-action of the measurement outcome y_k on the quantum state ρ_{k+1} as well as measurement imperfections and feedback delays, since these experimental issues were considered in the feedback law implemented in [11]. The relevance of the global stabilization results of arbitrary Fock states here established lies in the preparation of nonclassical field states that are robust with respect to environmental decoherence, which in turn is an important issue in quantum information applications [11]. Furthermore, in comparison to the Lyapunov function used in [11], the modifications proposed in this paper may lead to improvements in the speed of convergence towards the goal Fock state (see the simulation results in Section 4 and the conclusions in Section 6).

Loosely speaking, in [11], the control value u_k at each sample step k was chosen so as to minimize the conditional expectation of the Lyapunov function $V(\rho_k) = \mathrm{Tr}\left(d(\boldsymbol{N})\widehat{\rho}_{k+\tau}\right)$, where $\boldsymbol{N}$ is the photon-number operator, $d(n) = (n - \overline{n})^2$, $\overline{\rho} = |\overline{n}\rangle\langle\overline{n}|$ is the goal Fock state, τ is the control delay, and $\widehat{\rho}_{k+\tau}$ is a τ step(s) ahead estimate of the quantum state ρ_k. Bear in mind that, due to the delay τ, the control u_k computed at the sample step k will only be applied at $k+\tau$, thus justifying the role of the state estimator. However, in closed-loop, the difference between such V and its conditional expectation is not strictly positive: V does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control u is a smooth function of the state ρ. This cannot be the case here since u is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term $-\epsilon\sum_{n=0}^{\infty}(\langle n|\rho_k|n\rangle)^2$ to $V(\rho_k)$, where $\epsilon > 0$. This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis.

Moreover, contrarily to [1], the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator ρ_0 with a finite photon-number support ($\rho_0|n\rangle = 0$ for n large enough), the closed-loop trajectory $k \mapsto \rho_k$ also remains with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on ρ_0 is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The authors have provided in [9] a first mathematical convergence analysis of the Fock states feedback stabilization scheme described above under an ideal set-up, that is, by disregarding detection errors and control delays. This paper extends such analysis to more realistic experimental set-ups that are subject to measurement imperfections and feedbacks delays, as the ones reported in [11]. Such generalization is not trivial.

The paper is organized as follows. Section 2 presents a realistic Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [11], and precisely formulates the Fock states stabilization problem here treated (see Definition 1). Section 3 establishes the proposed solution to the control problem in two distinct parts. Firstly, Section 3.1 considers the particular case where the initial condition ρ_0 is a diagonal density operator (see Theorem 1). Only the main ideas of the convergence proof are outlined. The technical details are given in Section 5. Although such diagonal case is somewhat artificial and does not correspond to a practical physical context, it considerably simplifies the computations and the reasonings involved in general non-diagonal situations. The main result of the paper is presented in Section 3.2: the general solution is straightforwardly obtained from the diagonal case (Theorem 1) for ρ_0 belonging to a dense subset (see Theorem 2). The simulation results are exhibited in Section 4. The proof of some intermediate results and computations required in Sections 3 and 5 are presented in Appendix B–Appendix G. Finally, the concluding remarks are given in Section 6.

2. Realistic Markov model

The Fock states feedback stabilization scheme via single-photon corrections experimentally tested in [11] considered measurement imperfections and feedback delays. In order to take into account in the global stabilization results here established such experimental issues, as well as the back-action of the measurement outcome y_k on the quantum state ρ_{k+1}, one requires a realistic Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [11]. This is described in the sequel. Denote by $\mathcal{H}$ the separable complex Hilbert space $L_2(\mathbb{C})$ with orthonormal basis $\{|n\rangle, n \in \mathbb{N}\}$ of Fock states (photon-number). Hence, $\mathcal{H} = \{\sum_{n\in\mathbb{N}}\psi_n|n\rangle,\ (\psi_0, \psi_1, \ldots) \in l_2(\mathbb{C})\}$. Let $\mathrm{Tr}\,(\cdot) \in \mathbb{C}$ denote the (linear) trace function on the set of trace-class operators on $\mathcal{H}$, and let $\mathbb{D}$ be the set of all density operators on $\mathcal{H}$, that is, the set of trace-class, self-adjoint, non-negative operators on $\mathcal{H}$ with unit trace (see e.g. [10]). The sample step, corresponding to a sampling period around 100 μs, is indexed by $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, $u_k \in \{-1, 0, 1\}$ is the control, $\rho_k \in \mathbb{D}$ the quantum state, $y_k \in \{g, e\}$ the measurement outcome, $\tau \in \mathbb{N}$ is the control delay and

$$\chi_k = (\rho_k, \beta_k) \text{ with } \beta_k = (\beta_{k,1}, \ldots, \beta_{k,\tau}) = (u_{k-1}, \ldots, u_{k-\tau}) \tag{1}$$

is the extended state taking into account the delay of τ sample step(s) in the control u_k. When $\tau = 0$, one has $\chi_k = \rho_k$, that is, β_k is empty. The Markov model of the controlled microwave super-conducting cavity used in [11] admits the following structure. The extended state $\chi_{k+1} = (\rho_{k+1}, \beta_{k+1})$ at sample step $k+1$ is given as:

$$\begin{aligned}
\rho_{k+1} &= \begin{cases} \rho^g_{k+1}, & \text{when } y_k = g \text{ with probability } p_{g,k}, \\ \rho^e_{k+1}, & \text{when } y_k = e \text{ with probability } p_{e,k}, \end{cases} \\
\beta_{k+1} &= (\beta_{k+1,1}, \beta_{k+1,2}, \ldots, \beta_{k+1,\tau}) = (u_k, \beta_{k,1}, \ldots, \beta_{k,\tau-1}),
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
\rho^g_{k+1} &= \frac{(1-\delta)\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + \delta\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})}{\mathrm{Tr}\left((1-\delta)\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + \delta\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})\right)}, \\
\rho^e_{k+1} &= \frac{\delta\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + (1-\delta)\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})}{\mathrm{Tr}\left(\delta\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + (1-\delta)\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})\right)}.
\end{aligned} \tag{3}$$

The notations are as follows:

- $\delta \in [0, \frac{1}{2}[$ is the detection error rate;
- the measurements outcomes $y_k = g$ and $y_k = e$ occur with the respective probabilities[2]

$$\begin{aligned}
p_{g,k} &= \mathrm{Tr}\left((1-\delta)\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + \delta\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})\right), \\
p_{e,k} &= \mathrm{Tr}\left(\delta\boldsymbol{M}_g(u_{k-\tau})\rho_k\boldsymbol{M}_g^\dagger(u_{k-\tau}) + (1-\delta)\boldsymbol{M}_e(u_{k-\tau})\rho_k\boldsymbol{M}_e^\dagger(u_{k-\tau})\right) \\
&= 1 - p_{g,k};
\end{aligned} \tag{4}$$

[2] As usual in quantum physics, it is here assumed that the measurement outcome $y_k = y$ cannot occur when $p_{y,k} = 0$, for $y = g, e$.