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The ISS approach to the stability and robustness properties of nonautonomous systems with decomposable invariant sets: An overview

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ABSTRACT

This paper is an overview of recent developments in the Input-to-State Stability framework, dealing in particular with the extension of the classical concept to systems with multiple invariant sets and possibly evolving on Riemannian manifolds. Lyapunov-based characterizations of the properties are discussed as well as applications to the study of cascaded nonlinear systems.

1. Introduction

1.1. Notion and importance of multistability

The analysis of the stability and robustness properties of non-linear open dynamical systems exhibiting a variety of non-trivial dynamical behaviors – multiple equilibria, periodicity, almost-periodicity, chaos – has great importance to several scientific disciplines ranging from mechanics, electronics, and physics to biology and neuroscience. Indeed, the phenomenon known as multistability, i.e. the coexistence of multiple attractors (with possibly very diverse nature) in a system of differential equations with a given set of parameters, is frequently found in many real physical systems. A thorough survey about the different domains in which multistability occurs has been presented in [40].

Among the reasons which make the study of multistable systems appealing from the perspective of system and control theory are a number of properties exhibited by such systems, as in the following. Firstly, they possess a “memory” of past states and, as components of larger systems, can act as switches or underlie relaxation oscillators [20]. Secondly, as essential components of many biological systems, they display functional flexibility in response to various transitory stimuli and, furthermore, they play a crucial role in cell differentiation and in the maintenance of phenotypic differences in the absence of environmental distinctions [40]. Finally, again as components of larger networks, their dynamics might be at the very core of the mechanisms that entail information-processing or even intelligence [29]. The analysis of multistability thus appears as a key preliminary step towards what can be envisioned as a new kind of control engineering, mainly oriented to applications in biology and neuroscience.

1.2. Notion and importance of ISS

In this context, stability notions for nonlinear systems with respect to exogeneous input disturbances become key tools. In fact, they allow to analyze stability of interconnected systems in terms of input–output gains of individual subsystems and, at the same time, they provide quantitative estimates of how each sub-system reacts to exogenous disturbances [9]. In this direction, the Input-to-State Stability (ISS) approach [51,53,54] has extended the classical Lyapunov methods – traditionally used to establish internal stability properties – to systems with inputs and outputs, by exploiting energy-like functions in order to assess the stability and robustness of a system with respect to internal and external perturbations. We briefly recall here the classical notion of ISS together with a related notion called integral Input-to-State Stability (iISS) [10].

(*ISS*): The definition of ISS implies the qualitative property of having the state eventually trapped in a ball whose radius is

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proportional to the magnitude of the input, and thus represents a measure of performance in the qualitative analog of "finite $\mathcal{L}^\infty$ to $\mathcal{L}^\infty$ induced norm". In particular, a general nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad y = h(x(t)), \tag{1}$$

with state in $\mathbb{R}^n$, inputs in $\mathbb{R}^m$, and outputs in $\mathbb{R}^p$ is said to be *input-to-state stable* if it satisfies the following $\mathcal{L}^\infty \to \mathcal{L}^\infty$ property: there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that:

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\|u\|), \tag{2}$$

for all inputs $u(\cdot)$ and all initial conditions $x(0) \in \mathbb{R}^n$. In (2), we have denoted the Euclidean norm with $|\cdot|$ and the $\mathcal{L}^\infty$ norm of signal $u(\cdot)$ with $\|u\| := \sup_{t \ge 0} |u(t)|$.

(*iISS*): As a weaker but still very meaningful notion of stability, the Integral Input-to-State Stability (iISS) has been introduced in [50]. The definition of iISS implies the qualitative property of small overshoot when disturbances have finite energy and thus represents a measure of performance in the qualitative analog of "finite $\mathcal{H}^2$ norm". System (1) is said to be *iISS* if there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that:

$$|x(t)| \le \beta(|x(0)|, t) + \int_0^t \gamma(|u(s)|)\, ds, \tag{3}$$

for all inputs $u(\cdot)$ and all initial conditions $x(0) \in \mathbb{R}^n$. The iISS property has been presented in terms of asymptotic time-domain characterizations, Lyapunov dissipation inequalities, and variants of the $\mathcal{L}^2 \to \mathcal{L}^\infty$ estimate in references [10,8,11].

Applications: We are now going to mention a number of applications of the ISS property, and we refer the reader to [51] for an extensive presentation of the subject. Applications of the ISS property to the stabilization of classes of nonlinear systems [42], nonlinear cascades [49,52] and feedback interconnections [33,32]. Moreover, it has been shown in [49] that it is always possible for a GAS system to be refined – by means of feedback redesign – into a system which is ISS with respect to actuator disturbances. Applications of the iISS property are in the stabilization and disturbance attenuation of systems with bounded controls [34], nonlinear cascades [12], large-scale systems via decentralized output-feedback control [30], systems in block strict-feedback form via output regulation [28,31], and hybrid switched systems [36].

1.3. ISS and multistability: the "wrong" direction

All classical formulations of the ISS property and its related properties [51] characterize stability properties in a global setting with respect to a single equilibrium at the origin and for systems defined in Euclidean space. In other words, classical ISS implies such system to evolve on the "flat" Euclidean space and such equilibrium to be globally asymptotically stable in the absence of inputs. Interestingly enough, the classical definition of ISS allows in principle to formulate and characterize stability properties with respect to arbitrary compact invariant sets (and not simply equilibria), which automatically implies these arbitrary compact invariant sets to be both Lyapunov stable and globally attractive.[1] This requirement is applicable only to a class of compact invariant sets which consist of a single connected component [17]. Unfortunately, the invariant sets fail to consist of a single connected component in many behaviors of interest such as almost global stability [44], bistability, multistability, periodic oscillations, convergence to almost-periodic attractors and to chaotic attractors. It is important to recall that stability analysis of each invariant solution can be conducted locally by means of standard tools. Nevertheless, as mentioned before, a number of applications and open problems in many domains of interest (theoretical biology, electro-mechanical systems, etc.) call for a global analysis of stability for these invariant solutions.

In regard to the global analysis of multistable systems, several approaches are available in the literature. In addition to the first monograph on the subject [25], the papers [38,43] established the existence of Lyapunov functions for multistable systems evolving on a compact manifold. A modern approach is based on the aforementioned notion of almost global stability [44], that is convergence to an asymptotically stable equilibrium from all initial conditions except for those lying in a zero-measure set. The corresponding notion for nonautonomous systems is denoted as almost ISS [6]: the general nonlinear system (1) is said to be *almost ISS* with respect to an invariant compact set $\mathcal{A} \subset \mathbb{R}^n$ if $\mathcal{A}$ is locally asymptotically stable and there exists class-$\mathcal{K}_\infty$ function γ such that:

$$\forall u \ \forall \text{a.a. } \xi \in \mathbb{R}^n \quad \limsup_{t \to +\infty} |x(t, \xi, u)|_{\mathcal{A}} \le \gamma(\|u\|).$$

The key idea of this approach is to replace Lyapunov functions by suitable density functions and to impose a condition on the way these are propagated by the flow. The success of this approach has been validated both in terms of converse dual Lyapunov results fully characterizing the property [45], and in terms of software tools able to automatically find such density functions for certain classes of systems [41]. However, it has been shown in [5] that a fundamental limitation arises in the attempt of constructing density functions for a specific class of systems, such class encompassing the classical nonlinear pendulum model.

The difficulties in finding density functions pushed the research towards a complementary set of tools for the global analysis of stability and robustness in multistable systems [9]. The approach followed in [9] heavily relies on the stable and unstable manifold theory of dynamical systems and on their time-varying adaptation and, for this reason, provides a very fine structure to the stability properties of the invariant sets. In fact, the main result of the paper guarantees almost ISS for ultimately bounded multistable systems whose linearization at each unstable equilibrium has at least one eigenvalue with positive real part. Even though a possible extension of this result to multiperiodic systems could be conjured by drawing the theory of Floquet multipliers in, such result would still be not only very hard to test analytically but also merely a sufficient result, thus not providing a full characterization of the almost ISS property.

Moving from the almost ISS framework, the paper [17] has shown that the most natural way of conducting a global stability analysis for systems with multiple invariant sets is to relax the Lyapunov stability requirement rather than the global nature of the attractivity property. This intuition has led to a new line of research which starts from the characterizations of ISS for this class of systems in terms of usual Lyapunov-like inequalities [7], thus generalizing the classical ISS theory as well as the already-mentioned literature on autonomous flows evolving on compact manifolds [38]. This paper provides an overview of this current line of research and its related novel contributions. Furthermore, we would like to point out that the developed theory has already shown to be of great potential interest for applications in the domain of power systems [19,18] and robotics [26].

1.4. Notation

The Riemannian distance between two points x_1, x_2 of a Riemannian manifold M will be denoted with $\mathfrak{d}[x_1, x_2]$. We will

[1] Non-compact sets can somehow be tackled by considering output maps, and defining appropriate input-to-output stability properties. This is effective at least in the case of systems whose solutions are eventually trapped in a compact set (UBIBS systems) and for this case Lyapunov-like characterizations of the properties are possible [48,55].