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Zames–Falb multipliers for absolute stability: From O'Shea's contribution to convex searches

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ABSTRACT

Absolute stability attracted much attention in the 1960s. Several stability conditions for loops with slope-restricted nonlinearities were developed. Results such as the Circle Criterion and the Popov Criterion form part of the core curriculum for students of control. Moreover, the equivalence of results obtained by different techniques, specifically Lyapunov and Popov's stability theories, led to one of the most important results in control engineering: the KYP Lemma.

For Lurye[1] systems this work culminated in the class of multipliers proposed by O'Shea in 1966 and formalized by Zames and Falb in 1968. The superiority of this class was quickly and widely accepted. Nevertheless the result was ahead of its time as graphical techniques were preferred in the absence of readily available computer optimization. Its first systematic use as a stability criterion came 20 years after the initial proposal of the class. A further 20 years have been required to develop a proper understanding of the different techniques that can be used. In this long gestation some significant knowledge has been overlooked or forgotten. Most significantly, O'Shea's contribution and insight is no longer acknowledged; his papers are barely cited despite his original parameterization of the class.

This tutorial paper aims to provide a clear and comprehensive introduction to the topic from a user's viewpoint. We review the main results: the stability theory, the properties of the multipliers (including their phase properties, phase-equivalence results and the issues associated with causality), and convex searches. For clarity of exposition we restrict our attention to continuous time multipliers for single-input single-output results. Nevertheless we include several recent significant developments by the authors and others. We illustrate all these topics using an example proposed by O'Shea himself.

1. Introduction

A feedback interconnection of a linear system and a static nonlinearity is said to be *absolutely stable* if the interconnection is stable (in some sense) for every nonlinearity in a given class. The theory of absolute stability has occupied an important portion of the control theory literature due to its relevance to a variety of practical control/systems engineering problems. The absolute stability problem can be studied, broadly, from either the perspective of internal stability, or from that of input–output stability. The former, and perhaps more common, approach typically involves the search for the parameters of a proposed Lyapunov function which can be used to guarantee asymptotic stability of the origin for as large a class of nonlinearities as possible. The latter approach involves the use of transfer functions called *multipliers*. In their classical interpretation they are used to translate one nonlinear passivity-type problem into another linear, easier to solve, passivity-type problem. The aim, again, is to choose a multiplier within a predefined class of multipliers which allows input–output stability to be guaranteed for as large a class of nonlinearities as possible. In this paper, attention is focused on input–output stability from the perspective of passivity and in particular on the properties of the so-called Zames–Falb multipliers.

The multiplier approach attracted much attention from the control community in the 1960s. One reason for this was, without the computing power of today, researchers were able to glean a great deal about the absolute stability of a system purely from the properties of the linear part. In an early paper the concept of multiplier was used by Brockett and Willems [9] and the idea developed rapidly from this (see [58]). Despite this early promise and flurry of activity, probably the most widely known absolute

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[1] Also written as Lur'e or Lurie.

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stability tools today are the Circle and Popov Criteria (see [79,38]) which have become standard, in part due to their simplicity and in part due to their graphical interpretations. However, when a tighter description of the nonlinearity is available, these criteria are well-known to be conservative. In such cases, the use of more general multiplier methods can be useful and, in particular, the so-called Zames–Falb multipliers can often be used to improve predictions made about stability and performance of the interconnection.

Despite their moniker, Zames–Falb multipliers were actually discovered by O'Shea (his portrait is shown in Fig. 1) in [59,60]. While the treatment of O'Shea [59] was restricted to causal multipliers, the aim of [60] was to extend this definition to noncausal multipliers: "this modification allows greater freedom in the phase variation of $G(j\omega) + 1/k$ outside of the $\pm 90^\circ$ band". There were several correspondence items discussing these [88,85,23]. A rigorous and correct treatment was first given in the much-cited paper by Zames and Falb [89]. The contribution of O'Shea was fully acknowledged by all concerned at the time. As an example, Desoer and Vidyasagar [22] state that the "idea of using noncausal multipliers is due to O'Shea."

Fig. 1. R.P. O'Shea, reproduced with kind permission of [71].

However, the class of multipliers aroused little further interest for 20 years, until the proposal of Safonov and Wyetzner [65] for computer-aided search and the illustration by Megretski and Rantzer [54] of multiplier analysis embedded within the framework of IQCs. In these and subsequent papers the pioneering work of O'Shea was largely overlooked. The terminology "Zames–Falb multiplier" appears to have been coined by Chen and Wen [18,19] in their proposal for a convex search. This development, while rightly acknowledging the important work of Zames and Falb, has had an unfortunate consequence. Zames and Falb [89] focus on the relation of the nonlinearity to the monotone and bounded static nonlinearity; O'Shea's insights into the phase properties of the multipliers have been largely forgotten (with one notable exception: the discussion of Megretski [51] on phase limitation).

In this tutorial paper we re-examine Zames–Falb multipliers and, in particular, use an example of O'Shea [60] to discuss the phase properties of the Zames–Falb multipliers and how they can be used advantageously in the study of the absolute stability problem.

The remainder of the paper is structured as follows. In Section 2 we provide a brief motivating example explaining the significance of Zames–Falb multipliers, and in Section 3 we review the basics of the absolute stability problem and some approaches to its solution. In Section 4 we address at length an example previously discussed by O'Shea [60]. In particular we discuss how a number of input–output stability methods can be used for analysis. This section includes a comprehensive treatment of the application of a multiplier originally proposed by O'Shea. In Section 5 further properties of Zames–Falb multipliers are discussed and in Section 6 a brief review of start-of-the-art computational searches is given. Further developments of Zames–Falb multipliers are discussed in Section 7 and open questions considered in Section 8. Finally in Section 9 we conclude and point to some other recent developments in the use of Zames–Falb multipliers. While we emphasise the tutorial aspect of this overview, some mathematical formalism and machinery is inevitable; this is given in the appendix.

2. Motivating example

Remark 1. Several concepts in this section are formally defined in Section 3 and/or the Appendix.

Since saturation is a memoryless and slope restricted nonlinearity, the Zames–Falb multipliers can be used to study the stability/robust stability of systems involving saturation [34]. We shall illustrate such analysis with an anti-windup example [39] where robust stability is to be established [73,55]. $U(s)$ and $Y(s)$ are the Laplace transform of the plant's input and output, respectively.

Consider a plant with additive uncertainty

$$Y(s) = \left(G(s) + \frac{1}{\gamma}\Delta\right)U(s), \tag{1}$$

where $G(s)$ is the nominal SISO transfer function and Δ represents additive uncertainty with, for any bounded signal u,

$$\|\Delta u\|_2 \le \|u\|_2. \tag{2}$$

In the case where Δ is restricted to be a linear time invariant (LTI) system we may write this as the familiar $\mathbf{H}_\infty$ norm condition

$$\|\Delta\|_\infty \le 1. \tag{3}$$

Suppose the controller has the internal model control structure given by

$$U(s) = -Q(s)(Y(s) - G(s)U(s)). \tag{4}$$

and illustrated in Fig. 2.

The robustness of such controllers is discussed at length by Morari and Zafiriou [56]. Briefly, if both G and Q are stable, then it follows from a small gain argument that the loop is stable provided

$$\|Q\|_\infty < \gamma. \tag{5}$$

Suppose now there is saturation in the loop, as in Fig. 3. Since the saturation operator is in series with Δ, a similar small gain argument [73] says that the loop remains stable provided (5) is

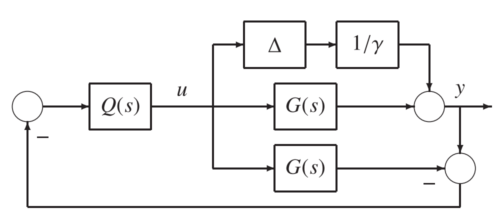

Fig. 2. Internal model control where the plant dynamics are assumed known save for an additive uncertainty.