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An approximate solution to state problem in coefficient-optimal-control problems ⋆

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Abstract: We consider an optimal control problem (OCP) for semilinear elliptic equations in inhomogeneous anisotropic media with discontinuous coefficients and solution (state) (DCS), as well as a non-linear component of state equations. Imperfect-contact matching condition is given at the inner boundary between media. We construct difference approximations for extremum problems and develop iterative processes for states in the OCPs. Numerical experiments are included.

Keywords: optimal control problem, elliptic equation, discontinuous coefficients and solution, iterative methods, numerical experiments

1. INTRODUCTION

In the paper we study an optimal control problem (OCP) described by semi-linear elliptic equations in inhomogeneous anisotropic media with discontinuous coefficients and solution (state) (DCS), as well as a non-linear component of state equations and imperfect-contact matching condition. This kind of problems for equations of mathematical physics (EMP) with matching boundary conditions of the imperfect-contact type naturally appears while modeling different processes in continuum mechanics, elasticity, heat transfer, diffusion. The coefficients and a solution to EMP admit discontinuities in the case when the domain is not homogeneous and consists of several parts with different properties, or the domain contains thin layers S with physical characteristics differing markedly from the basic media (see Samarskii and Andreev (1976), Samarskii (1989)). Assuming that these layers S are very thin and poorly permeable, we can describe the contact conditions by the relations:

$$p(x) = \left(\frac{\partial u}{\partial N_S}\right)^{-} = \left(\frac{\partial u}{\partial N_S}\right)^{+} = \theta(x)[u],\ x \in S,$$
$$\left(\frac{\partial u}{\partial N_S}\right)^{\pm} = \left(\sum_{\alpha=1}^{2} k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\cos(n, x_\alpha)\right)^{\pm},$$

where $[u] = u^+ - u^-$ is the jump of function u on S; $p(x)$ is a priori unknown substance (heat) flow through the elementary area; $\theta(x) \geq \theta_0 > 0$ is a given function, $S = \overline{\Omega}^- \cap \overline{\Omega}^+$ is the inner boundary of the media, $\Omega^- \cap \Omega^+ = \emptyset$, Ω^- and Ω^+ are some domains, so that $\Omega = \Omega^- \cup \Omega^+ \cup S$ is a bounded domain.

Problems for EMP with DCS are not as widely investigated. For instance, difference schemes for equations with discontinuous coefficients but continuous flow and solution (with perfect-contact matching condition) were developed by Samarskii and Andreev (1976), Samarskii (1989) for EMP with classical solutions of certain smoothness. The convergence of difference schemes for parabolic equations with DCS in classical formulation of the problems with sufficiently smooth solutions were studied Tsurko (2000), Tsurko (2005). Significant results for OCP, described by nonlinear EMP with DCS are obtained by Lubyshev (2012), Lubyshev, Manapova, and Fairuzov (2014), where difference scheme of extremum problems are constructed and investigated, estimates for approximations convergence rate w.r.t. state and functional are established, and approximations are regularized. Development of efficient numerical methods for solving finite-dimensional grid OCP is also essential. There are two stages to solving finite-dimensional grid OCP. First, we have the problem of constructing cost-effective, high-precision approximate methods of solving boundary value problems for EMP with DCS - problems for the state. Then we need to develop numerical algorithms for minimizing a cost functional, depending on a state of the system and a control.

In this work we develop approximate methods for solving boundary value problems of contact for elliptic equations with DCS, as well as a non-linear component of the state equations. Iterative processes with iterations on the inner boundary of the domain, where the coefficients and solution are discontinuous, reduce the initial problem to solution at each iteration to nonlinear boundary value problems in each contacting sub-domain of an integral domain. By applying iteration method with a parameter, we reduce the nonlinear difference problems in each of the sub-areas to linear difference problems. We implement iterative processes based on the upper relaxation method at each iteration. Computer experiments are included.

2. FORMULATION OF OCPS

Let $\Omega = \{r = (r_1, r_2) \in \mathbb{R}^2 : 0 \leq r_\alpha \leq l_\alpha, \alpha = 1, 2\}$ be a rectangle in $\mathbb{R}^2$ with a boundary $\partial\Omega = \Gamma$. The domain Ω is divided by the internal interface $\overline{S} = \{r_1 = \xi,\ 0 \leq r_2 \leq l_2\}$, where $0 < \xi < l_1$, into the left $\Omega_1 = \{0 < r_1 <$

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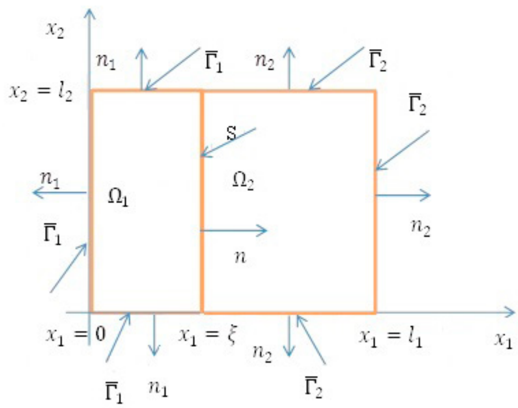

Fig. 1. The domain $\overline{\Omega}$.

$\xi,\ 0 < r_2 < l_2\}$ and right $\Omega_2 = \{\xi < r_1 < l_1,\ 0 < r_2 < l_2\}$ subdomains with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, while $\partial\Omega$ is the outer boundary of Ω. Let $\overline{\Gamma}_k$ denote the boundaries of Ω_k without S, $k = 1, 2$. Therefore $\partial\Omega_k = \overline{\Gamma}_k \cup S$, where Γ_k, $k = 1, 2$ are open nonempty subsets of $\partial\Omega_k$, $k = 1, 2$; and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega = \Gamma$. Let n_α, $\alpha = 1, 2$ denote the outward normal to the boundary $\partial\Omega_\alpha$ of Ω_α, $\alpha = 1, 2$. Let $n = n(x)$ be a unit normal to S at a point $x \in S$, directed, for example, so that n is the outward normal on S with respect to Ω_1; i.e., n is directed inside Ω_2. While formulating boundary value problems for states of control processes below, we assume that S is a straight line across which the coefficients and solutions of the problems are discontinuous, while being smooth within Ω_1 and Ω_2.

Consider the following Dirichlet problem for a semi-linear elliptic equation with DCS: Find a function $u(x)$, defined on $\overline{\Omega}$ that satisfies in Ω_1 and Ω_2 the equations:

$$-\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) + d(x)q(u) = f(x),\ x \in \Omega_1 \cup \Omega_2,$$

$$\text{and the conditions } u(x) = 0, \quad x \in \partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2, \quad (1)$$

$$\left[k_1(x)\frac{\partial u}{\partial x_1}\right] = 0,\ G(x) = \left(k_1(x)\frac{\partial u}{\partial x_1}\right) = \theta(x_2)[u],\ x \in S,$$

$$\text{where } u(x) = \begin{cases} u_1(x),\ x \in \Omega_1; \\ u_2(x),\ x \in \Omega_2, \end{cases} \quad q(\xi) = \begin{cases} q_1(\xi_1),\ \xi_1 \in \mathbb{R}; \\ q_2(\xi_2),\ \xi_2 \in \mathbb{R}, \end{cases}$$

$$k_\alpha(x), d(x), f(x) = \begin{cases} k_\alpha^{(1)}(x), d_1(x), f_1(x),\ x \in \Omega_1; \\ k_\alpha^{(2)}(x), d_2(x), f_2(x),\ x \in \Omega_2, \end{cases}$$

Here $[u] = u_2(x) - u_1(x)$ is the jump in $u(x)$ across S; $k_\alpha(x)$, $\alpha = 1, 2$, $d(x)$ are given functions that are defined variously in Ω_1 and Ω_2, and have a jump discontinuity on S; $q_\alpha(\xi_\alpha)$, $\alpha = 1, 2$, are given functions defined for $\xi_\alpha \in \mathbb{R}$, $\alpha = 1, 2$; $\theta(x_2)$, $x_2 \in S$, is a given function; and $g \equiv (f_1, f_2)$ is the control. The given functions are assumed to satisfy the following conditions: $k_\alpha(x) \in W_\infty^1(\Omega_1) \times W_\infty^1(\Omega_2)$, $\alpha = 1, 2$, $d(x) \in L_\infty(\Omega_1) \times L_\infty(\Omega_2)$, $\theta(x_2) \in L_\infty(S)$, $0 < \nu \le k_\alpha(x) \le \overline{\nu}$, $\alpha = 1, 2$, $0 \le d_0 \le d(x) \le \overline{d}_0$ for $x \in \Omega_1 \cup \Omega_2$ and $0 < \theta_0 \le \theta(x_2) \le \overline{\theta}_0$ for $x \in S$, where $\nu, \overline{\nu}, d_0, \overline{d}_0,\ \theta_0,\ \overline{\theta}_0$ are given constants; and the functions $q_\alpha(\zeta_\alpha)$ satisfy the conditions: $q_\alpha(0) = 0$, $0 < q_0 \le \left(q_\alpha(\zeta_\alpha) - q_\alpha(\overline{\zeta}_\alpha)\right)/(\zeta_\alpha - \overline{\zeta}_\alpha) \le L < \infty$, for all $\zeta_\alpha,\ \overline{\zeta}_\alpha \in \mathbb{R},\ \zeta_\alpha \ne \overline{\zeta}_\alpha$.

The set of admissible controls is defined as $U = \prod_{\alpha=1}^{2} U_\alpha \subset H = L_2(\Omega_1) \times L_2(\Omega_2)$, $U_\alpha = \{g_\alpha = f_\alpha \in L_2(\Omega_\alpha):\ 0 < g_0 \le g_\alpha(x) \le \overline{g}_0$ a.e. on $\Omega_\alpha\}$, $\alpha = 1, 2$, where g_0 and $\overline{g}_0$ are given constants and a.e. means almost everywhere.

The cost functional $J:\ U \to \mathbb{R}^1$ is defined as

$$g \to J(g) = \int_{\Omega_1} \left|u(r_1, r_2; g) - u_0^{(1)}(r)\right|^2 d\Omega_1, \qquad (2)$$

where $u_0^{(1)} \in W_2^1(\Omega_1)$ is a given function.

The optimal control problem is to minimize the cost functional (2) over the solutions $u(r; g)$ of problem (1), corresponding to all admissible controls $g = (f_1, f_2) \in U$.

Let $\overset{\circ}{\Gamma}_k$ be a portion of the boundary $\partial\Omega_k$. Denote by $W_2^1\left(\Omega_k; \overset{\circ}{\Gamma}_k\right)$ the closed subspace of $W_2^1(\Omega_k)$ in which the set of all functions from $C^1(\overline{\Omega}_k)$ vanishing near $\overset{\circ}{\Gamma}_k \subset \partial\Omega_k$, $k = 1, 2$ in a dense set. We introduce the space $\overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)}) = \{u = (u_1, u_2) \in W_2^1(\Omega_1; \Gamma_1) \times W_2^1(\Omega_2; \Gamma_2)\}$ with the norm $\|u\|^2_{\overset{\circ}{V}_{\Gamma_1,\Gamma_2}} = \sum_{k=1}^{2} \int_{\Omega_k} \sum_{\alpha=1}^{2} \left(\frac{\partial u_k}{\partial x_\alpha}\right)^2 d\Omega_k + \int_S [u]^2\, dS.$

The solution of direct problem (1) with a fixed control $g \in U$ is a function $u \in \overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)})$ satisfying the identity:

$$\int_{\Omega_1\cup\Omega_2} \left[\sum_{\alpha=1}^{2} k_\alpha \frac{\partial u}{\partial x_\alpha}\frac{\partial \vartheta}{\partial x_\alpha} + d\, q(u)\, \vartheta\right] d\Omega_0$$

$$+ \int_S \theta\, [u][\vartheta] dS = \int_{\Omega_1\cup\Omega_2} f\, \vartheta\, d\Omega_0, \quad \forall \vartheta \in \overset{\circ}{V}_{\Gamma_1,\Gamma_2}(\Omega^{(1,2)}).$$

3. DIFFERENCE APPROXIMATION OF OCPS

We construct approximations of the problems based on the grid method (see Samarskii et al.). To approximate problems (1)-(2), we need some grids on $[0, l_\alpha]$, $\alpha = 1, 2$, and in $\overline{\Omega}$, the inner products, norms, and seminorms of grid functions defined on various grids and corresponding grid spaces (see Manapova and Lubyshev (2014)).

Optimal control problems (1)-(2) are associated with the following difference approximations: minimize the grid functional

$$J_h(\Phi_h) = \sum_{\overline{\omega}^{(1)}} \left|y(\Phi_h) - u_{0h}^{(1)}\right|^2 \hbar_1\hbar_2 = \|y(\Phi_h) - u_{0h}^{(1)}\|^2_{L_2(\overline{\omega}^{(1)})},$$

provided that the grid function $y \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$, which is the solution of the difference boundary value problem for problem (1), satisfies, for any grid function $v \in \overset{\circ}{V}_{\gamma^{(1)}\gamma^{(2)}}(\overline{\omega}^{(1,2)})$, the summation identity $Q_h(y, v) =$

$$\sum_{\alpha=1}^{2}\Bigg(\sum_{\omega_1^{(\alpha)+}\times\omega_2} a_{1h}^{(\alpha)} y_{\alpha\overline{x}_1} v_{\alpha\overline{x}_1} h_1 h_2 + \sum_{\omega_1^{(\alpha)}\times\omega_2^+} a_{2h}^{(\alpha)} y_{\alpha\overline{x}_2} v_{\alpha\overline{x}_2} h_1 h_2$$

$$+\frac{1}{2}\sum_{\gamma_S} a_{2h}^{(\alpha)} y_{\alpha\overline{x}_2} v_{\alpha\overline{x}_2} h_1 h_2 + \sum_{\omega^{(\alpha)}} d_{\alpha h}\, q_\alpha(y_\alpha)\, v_\alpha\, h_1 h_2$$