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One estimation of the stability defect of sets in an approach game problem ⋆

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Abstract: A game problem of the approach to a compact target set at a fixed termination time is studied. We investigate the question of estimating the stability defect of a set in the space of game positions, which is weakly invariant with respect to a finite set of unification differential inclusions.

Keywords: Differential games, control theory, stable bridge, stability defect.

1. INTRODUCTION

We consider a conflict-control nonlinear system on a finite time interval and study a game problem of approach of the system to a target compact set at a fixed time. The main subject of the research is the notion of the stability defect of sets in the space of game positions of the system, earlier introduced and investigated by Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011). This notion was introduced in order to extend the stability property and due to the fact that quite often in the process of constructing stable bridges one can get sets not possessing the stability property. The stability property is a property of weak invariance of a set in the space of positions with respect to some set of differential inclusions related to the dynamics of the system. These sets of inclusions can be different, but they identify the same sets in the space of game positions, which are stable bridges. The formulation of the stability property based on unification turned out to be convenient for extending the stability property. It was proposed in researches of Krasovskii (1976), and Krasovskii (1977). In particular, unification definitions of the stability property presented in the infinitesimal form appeared to be effective, see Guseinov, Subbotin, and Ushakov (1985)

We note that unification sets of differential inclusions, expressed in unification definitions of the stability property and used in this paper, are infinite. For any non-trivial system it is impossible to realistically check, whether a set in the space of positions is stable bridge. This check can be carried out for some relatively simple conflict-control systems, e.g. for systems having a simple Hamiltonian, by the fact that the unification set can be replaced with a finite subset being equivalent in terms of the stability property.

For arbitrary conflict-control systems with complicated dynamics following problem becomes relevant. Suppose that one has selected a finite subset of unification set of differential inclusions. Furthermore, there is a set constructed in the space of positions, that is weakly invariant with respect to the subset. It is required to assess to what extent does the set possess the stability property, i.e. in what extent is it weakly invariant with respect to the whole unification set. In other words, it is required to estimate an upper bound of value of the stability defect of the set. Current paper is dedicated to derivation of one of these assessments.

2. SETTING OF THE APPROACH GAME PROBLEM

Let us consider a conflict-control system whose behavior on the time interval $[t_0, \vartheta]$ ($t_0 < \vartheta < \infty$) is described by the differential equation

$$\frac{dx}{dt} = f(t, x, u, v), \quad x(t_0) = x^0, \quad u \in P, \quad v \in Q. \quad (1)$$

Here, x is the m-dimensional state vector of the system; u is the control of the first player; v is the control of the second player; and P and Q are compact sets in the Euclidean spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

We assume that the right-hand side of equation (1) satisfies the following conditions.

Condition A. The function $f(t, x, u, v)$ is given and continuous on $[t_0, \vartheta] \times \mathbb{R}^m \times P \times Q$ and, for any bounded closed domain $D \subset [t_0, \vartheta] \times \mathbb{R}^m$, there exists a constant $L = L(D) \in (0, \infty)$ such that

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$$\|f(t, x^{(1)}, u, v) - f(t, x^{(2)}, u, v)\| \leqslant L\|x^{(1)} - x^{(2)}\|,$$
$$(t, x^{(i)}, u, v) \in D \times P \times Q, \quad i = 1, 2.$$

Here, $\|f\|$ stands for the norm of the vector f in the Euclidean space.

Condition B. There exists a constant $\gamma \in (0, \infty)$ such that

$$\|f(t, x, u, v)\| \leqslant \gamma(1 + \|x\|),$$
$$(t, x, u, v) \in [t_0, \vartheta] \times \mathbb{R}^m \times P \times Q.$$

In the approach game problem facing the first player, it is required to ensure that the phase vector $x(\vartheta)$ of equation (1) hits a given compact set $M \subset \mathbb{R}^m$. The solution of the approach problem is required to be in the class of positional control procedures of the first player, see Krasovskii, and Subbotin (1974).

The dual problem to the problem formulated above, is an evasion problem facing the second player, in which it is required to ensure the evasion of the phase vector $x(\vartheta)$ of equation (1) from some ε-neighborhood of the compact set M. The solution of the evasion problem is required to be in the class of counter-positional control procedures with a guide of the second player, see Krasovskii, and Subbotin (1974).

A differential approach-evasion game is composed of the approach problem and the evasion problem. According to Guseinov, Subbotin, and Ushakov (1985), the game has the following alternative: there exists a closed set $W^0 \subset [t_0, \vartheta] \times \mathbb{R}^m$ such that the approach problem is solvable for all initial positions $(t_*, x_*) \in W^0$ and the evasion problem is solvable for all initial positions $(t_*, x_*) \in \big([t_0, \vartheta] \times \mathbb{R}^m\big) \setminus W^0$. The set W^0 plays a decisive role in solving the differential approach-evasion game. For initial positions $(t_*, x_*) \in W^0$ a resolving positional control procedure of the first player can be implemented as a positional control procedure with a guide, that directs the phase vector $x(t)$ of equation (1) towards the guide moving in the set W^0. It is known that the set W^0 has following very important property: W^0 is the maximal u-stable bridge, see Krasovskii, and Subbotin (1974). This property forms the basis of algorithms for the approximate calculation of W^0. Some studies use algorithms for the approximate calculation of W^0 based on unification constructions or their modifications, see Tarasev, and Ushakov (1987) and Tarasev, Ushakov, and Khripunov (1987).

Let us formulate a definition of the stability property of sets contained in $[t_0, \vartheta] \times \mathbb{R}^m$ based on unification set of differential inclusions in the infinitesimal form. To describe the unification set we introduce Hamiltonian function H of equation (1) and a set of multivalued mappings $\mathcal{L}$.

Assume that

$$H(t, x, l) = \max_{u \in P} \min_{v \in Q} \langle l, f(t, x, u, v) \rangle,$$
$$(t, x, l) \in [t_0, \vartheta] \times \mathbb{R}^m \times \mathbb{R}^m,$$

is Hamiltonian function of equation (1), where $\langle l, f \rangle$ is the inner product of the vectors l and f from $\mathbb{R}^m$.

Taking into account Condition B and the definition of the set W^0, we conclude that one can find a sufficiently large bounded closed domain D in $[t_0, \vartheta] \times \mathbb{R}^m$ that contains the set W^0 and all motions $x(t)$ coming to some ε-neighborhood of the set M, i.e. $(t, x(t)) \in D$ when $t \in [t_0, \vartheta]$. Hereinafter D is fixed.

We choose $R \in (0, \infty)$ to be so large that

$$r = \max_{(t,x,l) \in D \times S} |H(t, x, l)| < R,$$

where $S = \big\{ l \in \mathbb{R}^m : \|l\| = 1 \big\}$.

We follow Krasovskii (1976) and Krasovskii (1977), and introduce following sets:

$$G = B(\mathbf{0}; R), \quad \Pi_l(t, x) = \big\{ f \in \mathbb{R}^m \colon \langle l, f \rangle \leqslant H(t, x, l) \big\},$$
$$F_l(t, x) = \Pi_l(t, x) \cap G, (t, x, l) \in D \times S.$$

Here $B(\mathbf{0}; R)$ is the closed ball in $\mathbb{R}^m$ with center at $\mathbf{0}$ and radius R.

Thus, the sets $F_l(t, x)$ are the spherical segments in the space $\mathbb{R}^m$, that don't degenerate for any $(t, x, l) \in D \times S$, i.e. sets have a non-empty interior.

Let us define the set $\mathcal{L}$ as a family of multivalued mappings $(t, x) \mapsto F_l(t, x)$ defined on D and corresponding to vectors $l \in S$. Obviously, the set $\mathcal{L}$ is uncountable.

We consider

$$\overrightarrow{D} W(t_*, x_*) = \Big\{ d \in \mathbb{R}^m \colon d = \lim_{k \to \infty} (t_k - t_*)^{-1} (w_k - x_*),$$
$$\{(t_k, w_k)\} \text{ is a sequence in } W,$$
$$t_k \downarrow t_* \text{ when } k \to \infty, \ \lim_{k \to \infty} w_k = x_* \Big\}.$$

$\overrightarrow{D} W(t_*, x_*)$ is the contingent derivative of the multivalued mapping $t \mapsto W(t) = \big\{ x \in \mathbb{R}^m : (t, x) \in W \big\}$ at the point $(t_*, x_*) \in W$, $t_* \in [t_0, \vartheta)$, see Guseinov, Subbotin, and Ushakov (1985).

Definition 1. A non-empty closed set $W \subset D$ is a u-stable bridge in the approach game problem at time ϑ if and only if:

(1) $W(\vartheta) \subseteq M$;
(2) $\overrightarrow{D} W(t_*, x_*) \cap F_l(t_*, x_*) \neq \varnothing, \quad t_* \in [t_0, \vartheta),$ $(t_*, x_*, l) \in \partial W \times S.$

The definition 1 is the definition of the stability property in the infinitesimal form. It embeds the notion of derivatives into theory of differential games. The definition turned out to be useful in identifying the various properties of stable bridges, see Ushakov et al. (2010). Additionally, it is helpful in the formation of new concepts and constructions in the theory of differential games, see Ushakov, and Latushkin (2006), Ushakov, and Uspenskii (2010), and Ushakov, and Malev (2011).

3. STABILITY DEFECT OF SETS IN THE SPACE OF GAME POSITIONS

In this section, we give a definition of the stability defect of a set $W^* \subset D$. We assume that $W^*(\vartheta) = M$ and W^* has the continuity property: if $t_0 \leqslant t_* < t^* \leqslant \vartheta$ and $W^*(t_*) \neq \varnothing$ then $W^*(t^*) \neq \varnothing$.

Moreover, strengthening the continuity property of the set W^*, we assume that the following condition is satisfied.

Condition C.

$$d\big(W^*(t_*), W^*(t^*)\big) \leqslant R(t^* - t_*), \quad t_0 \leqslant t_* < t^* \leqslant \vartheta.$$