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On the Maximum Displacement of a System of Interacting Point Masses Along a Straight Line with Dry Friction ⋆

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Abstract: The paper investigates the motion of the system, consisting of three or more identical point masses, along a straight line with dry friction. The motion occurs when the configuration changes due to the forces acting between the masses. An optimal control problem is solved to maximize the displacement of the system for a fixed time with both initial and terminal states of the system having the same configuration and being at rest. No constraints are imposed on the interaction forces. Nonuniqueness of optimal solution is proved and an optimal solution is constructed such that the distance between any two points of the system is less than a given positive constant over all time of the motion.

Keywords: locomotion; vibration-driven systems; optimal control.

1. INTRODUCTION

The paper continues investigations dealing with the motion of systems with variable configuration along a horizontal straight line with dry friction. The interaction forces act between the bodies that compose the system. The forces change the distances between the bodies, the velocities of the bodies, and, consequently, the friction forces applied to them, which are external forces with respect to the system. Each body is modelled as a point mass, friction forces obeys Coulumb's law, and the normal force acting on each mass is constant (the weight of the system is not redistributed between the bodies). Anisotropic and isotropic friction is considered. In the latter case, the system has no advantage for moving one or another direction, and the motion in the desired direction is implemented by choosing a proper control forces of interaction and the parameters of the system.

The motion of a two-body system along a straight line is considered by Chernousko (2002, 2011), and Wagner et al. (2013). Chernousko (2002, 2011) solved the optimization problem for the motion of a system consisting of two bodies with different masses along a line with dry, in particular, isotropic, friction. The distance between the bodies and the velocities of both bodies are supposed to be periodic functions of time, and the average velocity of the system is maximized. The body with larger mass is supposed never to move backward, the interaction force is assumed to be piecewise constant in paper (Chernousko (2002)), and the velocity of the relative motion of the bodies is assumed to be piecewise constant in paper (Chernousko (2011)). The optimal parameters of the system and of the control laws are found. In both papers it is established that the body with smaller mass always has interval of the backward motion.

Paper (Wagner et al. (2013)) also deals with the motion of the two-body system along the line with isotropic dry friction. The distance between the bodies is supposed to change periodically so that there is one interval of moving away of the bodies and one interval of moving each to another. It is proved that the system can travel along the line iff two conditions are fulfilled. First, the kinetic dry friction forces of two bodies must be different, and, second, the distance between the bodies as a function of time must not have a symmetry with respect to the time instant when the distance between the body is maximal over the time interval between two states with minimal distance between the bodies.

Papers by Bolotnik et al. (2009, 2011) investigate the motion of the system of $n \geq 3$ identical bodies along a straight line with small friction, in particular, for the case of small dry isotropic friction. The distance between each pair of the neighboring bodies changes in accordance with the same function of time, to within a time delay which is constant from one pair of adjacent bodies to the next one. This function is assumed to be piecewise linear in paper (Bolotnik et al. (2009)), and piecewise quadratic in paper (Bolotnik et al. (2011)). The parameters of the system are obtained allowing the system to get started from the state of rest and the velocity of steady state motion is found depending on the parameters of the system. In the motion with small friction all bodies have intervals of backward motion.

The present paper deals with the motion of a system of three or more identical bodies along the straight line with dry isotropic friction, the smallness of the friction is not assumed. The system can move so that all bodies never move backward. An optimal control problem is solved to

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maximize the distance that the system travels for a given time. It is assumed that the forces of interaction are not constrained and allow jumps in the velocities of the bodies.

2. OPTIMAL CONTROL PROBLEM STATEMENT

Let us consider a system consisting of $n \geq 3$ point masses, lying on a horizontal straight line. The masses of the points are supposed to be identical, $m_i = m$, $i = 1, ..., n$, dry friction forces are exerted by the straight line on the point masses. The interaction forces acting between the adjacent points are taken as the control variables, no constraints being imposed on their values and, particularly, the control forces may be δ-functions, so that the velocities of the points can change in jump instantly. Let x_i be the coordinate of the i-th point along the line of motion, and let v_i and F_i be the velocity of the ith point and the dry friction force acting on it. Let f_i be the control force applied by point i to point $i+1$.

The motion of the system of point masses is governed by the equation

$$\begin{aligned} &\dot{x}_i = v_i, \\ &m\dot{v}_i = f_{i-1} - f_i + F_i, \end{aligned} \qquad i = 1, ..., n. \tag{1}$$

By definition we assume $f_0 = f_n = 0$. The dry friction forces obey the relations

$$F_i = \begin{cases} -kmg\ \mathrm{sgn} v_i, & \text{if } v_i \neq 0, \\ -f_{i-1} + f_i, & \text{if } v_i = 0 \text{ and } |f_{i-1} - f_i| \leq kmg, \\ -kmg\ \mathrm{sgn}(f_{i-1} - f_i), & \text{if } v_i = 0, |f_{i-1} - f_i| > kmg, \end{cases} \tag{2}$$

Here k is the dry friction coefficient and g is the acceleration due to gravity.

Let at the initial time instant $t = 0$ all points of the system be at rest and let the positions of all points be the same (this initial position of the system on the straight line assumed to coincide with the origin of coordinates):

$$x_i(0) = 0, \quad v_i(0) = 0, \quad i = 1, ..., n. \tag{3}$$

Let the duration of the motion of the system be fixed, $t \in [0, T]$. The motions of the system are considered which lead all points of the system to the same final position on the straight line with zero velocities:

$$x_i(T) = x_1(T),\ i = 2, ..., n, \quad v_i(T) = 0,\ i = 1, ..., n. \tag{4}$$

Denote by x and v the coordinate of the center of mass of the system and its velocity:

$$x = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad v = \frac{1}{n}\sum_{i=1}^{n} v_i. \tag{5}$$

The problem of maximum displacement for this system is stated as follows.

Problem 1. Find a motion of the point masses system obeying the relations (1), (2) with unbounded control forces f_i, which transfers the system from the state (3) to the state (4) and maximizes the displacement of the system

$$x(T) \to \max. \tag{6}$$

3. AUXILIARY OPTIMAL CONTROL PROBLEM FOR THE MOTION OF THE CENTER OF MASS

Equations (1) yield the following equation of motion for the center of mass of the system

$$\dot{v} = \frac{1}{nm}\sum_{i=1}^{n} F_i. \tag{7}$$

Suppose that the motion of the system $x_i(t)$, $i = 1, ..., n$, together with friction forces $F_i(t)$ satisfying the relations (2), are given, and the equality (7) holds in some time interval. Then the control forces generating such a motion have the form

$$f_i = \sum_{j=1}^{i} F_j - m\sum_{j=1}^{i} \dot{v}_j, \qquad i = 1, ..., n-1. \tag{8}$$

An auxiliary problem of maximum displacement for the center of mass is posed as follows.

Problem 2. Find a motion of the system of point masses obeying the relations (1), (2) with unbounded control forces f_i such that the velocity of the center of mass is equal to zero at the initial and terminal time instants and the displacement of the center of mass is a maximum:

$$v(0) = v(T) = 0, \quad x(0) = 0, \tag{9}$$

$$x(T) \to \max. \tag{10}$$

Note that the boundary conditions of Problem 1, according to which the system is in a state of rest in the initial and terminal positions and all points perform the same displacement, are not required for Problem 2. Hence, the set of admissible motions in Problem 1 is a subset of the set of admissible motions in Problem 2 and, therefore, the maximum value of $x(t)$ in Problem 2 in greater than or equal to its maximum value in Problem 1.

With the velocity of the center of mass given, the velocities of the points of the system can changed instantly so that they acquire any values satisfying the second equality of (5). Let at a time instant t the velocities v_i change in jump so that $v_i(t+0) = v_i(t-0) + \Delta v_i$. By virtue of (7) and boundedness of friction forces, the velocity of the center of mass is continuous, $v(t+0) = v(t-0)$, and the increments of the velocities of the points, under condition (5), obey the relation

$$\sum_{i=1}^{n} \Delta v_i = 0. \tag{11}$$

The control forces of interaction that cause the increments are calculated as

$$f_i(t) = -m\delta(t)\sum_{j=1}^{i} \Delta v_j, \qquad i = 1, ..., n-1, \tag{12}$$

where $\delta(t)$ is Dirac's delta function. Hereby, it is possible to change instantly the velocity of the points in any way consistent with (11), using the control forces (12).

Let us construct the optimal motion of the center of mass $x(t)$ in Problem 2. The respective motion of the points of the system are supposed to exist and will be described in what follows. Let the velocity of the center of mass be