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Constrained Primal-Dual Dynamics for QCQP with Applications to Economic Dispatch

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Abstract: We develop concise primal-dual dynamics for a class of Quadratically Constrained Quadratic Programming problems in power system optimization. Using a constrained Lagrangian reformulation of the problem and the classical stability result of Lyapunov, we establish the asymptotic convergence of the primal-dual dynamics. We demonstrate the efficiency of the proposed method on an economic power dispatch problem with transmission losses and we suggest a neural network architecture for real-time optimization.

Keywords: Economic Dispatch, Primal-Dual Dynamics, Constrained Optimization , Quadratically Constrained Quadratic Programming

1. INTRODUCTION

Quadratically constrained quadratic programming (QCQP) is widely applied in engineering. Recent applications include among others relay coordination of power system (Papaspiliotopoulos et al., 2017), power flow optimization (Bose et al., 2015) and economic dispatch (Zhong et al., 2013). In general, the QCQP problem is nonconvex and NP-hard. However, in certain instances, it is possible to exploit the physical structure of the problem to develop efficient and polynomial-time algorithms (Bose et al., 2015; Konar and Sidiropoulos, 2015). In these cases, iterative solution methods such as interior point method (Torres and Quintana, 1998), Hopfield neural networks (Su and Lin, 2000) and alternating direction method of multipliers Huang and Sidiropoulos (2016) have been employed.

Recently, there has been a surge in continuous-time gradient-based methods for fast computation and real-time implementation of nonlinear programming problems(Arrow et al., 1958; Feijer and Paganini, 2010; Cherukuri et al., 2016). Such gradient dynamics have been applied in active loss minization Ma and Elia (2013), in congestion control applications Feijer and Paganini (2010) and in load sharing Yi et al. (2015). Circuit implementation have also been considered in Costantini et al. (2008) Levenson and Adegbege (2016). This class of methods offers light-weight and efficient algorithms that can be implemented in real-time with little or no storage requirements.

Motivated by these previous results, we develop in this paper concise primal-dual gradient dynamics for a class of convex QCQP problems. The proposed method extends the results of Costantini et al. (2008) to cases where the constraints are not affine in the variable and where the nonlinearity describing the bound constraints is not necessarily passive. Using a bound-Lagrangian reformulation of the problem, we develop a compact gradient-based algorithm to seek the saddle-point of the Lagrangian function which in turn solves the original QCQP. The efficiency of the proposed algorithm is tested on an economic dispatch problem incorporating transmission losses.

Economic dispatch is an important energy management problem which deals with power mismatch, fuel cost economy, and transmission losses reduction. The traditional economic dispatch problem can be formulated as a QCQP problem. We derive sufficient condition for which the system is asymptotically stable and globally convergent to a unique solution.

We organize the remainder of the paper as follows: In section 2, we define the class of QCQP problems under consideration. In section 3, we develop compact primal-dual dynamics for seeking the saddle-point solution of the Lagrangian reformulation of the QCQP problem. We also provide convergence analysis of the primal-dual algorithm using concepts of Lyapunov stability and we suggest neural network architecture for efficient implementation of the QCQP. In section 4, we consider a simulation example using economic power dispatch problem incorporating transmission losses.

The notation adopted throughout the paper is standard.

2. PROBLEM FORMULATION

We consider quadratically constrained quadratic programming problem of the form:

$$\underset{x}{\text{minimize}} \quad f(x) \tag{1a}$$

$$\text{subject to } g_i(x) = 0, \; i = 1, \cdots, m, \tag{1b}$$

$$x \in \mathcal{X}, \tag{1c}$$

with

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$$f(x) = \frac{1}{2}x^T H_0 x + {q_0}^T x + r_0, \tag{2}$$

$$g_i(x) = \frac{1}{2}x^T H_i x + {q_i}^T x + r_i, i = 1, \cdots, m, \tag{3}$$

$$\mathcal{X} = \{x \mid l_i \leq x_i \leq u_i,\ i = 1, \cdots, n\}, \tag{4}$$

where $H_i \in \mathbb{R}^{n\times n}$, $h_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}, i = 0, \cdots, m$ are the problem data, and $l \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ with $0 \leq l < u$ are respectively the lower and the upper bound on the variable $x \in \mathbb{R}^n$ to be optimized. We assume that H_0 is symmetric positive definite (i.e. $H_0 = H_0^T > 0$) and that H_i is symmetric positive semi-definite (i.e. $H_i = H_i^T \geq 0;\ i = 1...m$) such that problem (1) is strictly convex and solvable in polynomial time (Boyd and Vandenberghe, 2004).

We define a Lagrangian function $\mathcal{L}(x, \lambda, \overline{\mu}, \underline{\mu})$ in terms of x and the multipliers λ(indefinite) $\in \mathbb{R}^m$, $0 \leq \underline{\mu} \in \mathbb{R}^n$ and $0 \leq \overline{\mu} \in \mathbb{R}^n$ for problem (1) as follows:

$$\mathcal{L}(\cdot) = f(x) + \sum_{i=0}^{m} \lambda_i g_i(x) + \underline{\mu}^T(l - x) + \overline{\mu}^T(x - u) \tag{5a}$$

$$= \frac{1}{2}x^T H(\lambda)x + q(\lambda)^T x + r(\lambda) + \mu^T(Ax - b), \tag{5b}$$

where

$$H(\lambda) = H_0 + \sum_{i=0}^{m} \lambda_i H_i,\ q(\lambda) = q_0 + \sum_{i=0}^{m} \lambda_i q_i,$$

$$r(\lambda) = r_0 + \sum_{i=0}^{m} \lambda_i r_i,\ \mu = \begin{bmatrix} \overline{\mu} \\ \underline{\mu} \end{bmatrix}, A = \begin{bmatrix} -I \\ I \end{bmatrix} \text{ and } b = \begin{bmatrix} -l \\ u \end{bmatrix}.$$

By the well-known saddle-point theorem (Boyd and Vandenberghe, 2004), a vector x^* solves problem (1) if there exist multipliers λ^*, μ^* such that together with x^*, the point (x^*, λ^*, μ^*) is a saddle-point solution of (5) i.e. the following inequality holds:

$$\mathcal{L}(x^*, \lambda, \mu) \leq \mathcal{L}(x^*, \lambda^*, \mu^*) \leq \mathcal{L}(x, \lambda^*, \mu^*). \tag{6}$$

The vectors x^* and (λ^*, μ^*) are said to be primal optimal and dual optimal with no duality gap if the following Karush-Kuhn-Tucker (KKT) optimality condition holds:

$$H(\lambda)x + q(\lambda) + \overline{\mu} - \underline{\mu} = 0, \tag{7a}$$

$$\frac{1}{2}x^T H_i x + {q_0}^T x + r_i = 0;\ i = 1, \cdots, m, \tag{7b}$$

$$\underline{\mu} \geq 0, \quad \underline{\mu}^T(l - x) = 0, \tag{7c}$$

$$\overline{\mu} \geq 0, \quad \overline{\mu}^T(x - u) = 0, \tag{7d}$$

$$(l - x) \leq 0,\ (x - u) \leq 0. \tag{7e}$$

In what follows, we develop three primal-dual dynamics that enforce the KKT optimality condition (7) at equilibrium and in turn provide the optimal solution to problem (1). To ensure that such solutions always exist, we make the following assumption.

Assumption 1. (Slater Condition) There exists x^* such that $l_i < x_i^* < u_i,\ i = 1, \cdots, n$ and $g_i(x^*) = 0,\ i = 1, \cdots, m$.

3. PRIMAL-DUAL DYNAMICS

A natural primal-dual dynamical system for seeking the saddle-point solution of (5) can be expressed as

$$\dot{x} = -K\nabla_x \mathcal{L} = -K\left(H(\lambda)x + q(\lambda) + \overline{\mu} - \underline{\mu}\right), \tag{8a}$$

$$\dot{\lambda}_i = \Gamma_i \nabla_{\lambda_i} \mathcal{L} = \Gamma_i \left(\frac{1}{2}x^T H_i x + q_i^T x + r_i\right), \tag{8b}$$

$$\dot{\underline{\mu}} = \Gamma_{\underline{\mu}}[\nabla_{\underline{\mu}} \mathcal{L}]^+_{\underline{\mu}} = \Gamma_{\underline{\mu}}[(l - x)]^+_{\underline{\mu}}, \tag{8c}$$

$$\dot{\overline{\mu}} = \Gamma_{\overline{\mu}}[\nabla_{\overline{\mu}} \mathcal{L}]^+_{\overline{\mu}} = \Gamma_{\overline{\mu}}[(x - u)]^+_{\overline{\mu}} \tag{8d}$$

for $i = 1, \cdots, m$, where $\nabla_x \mathcal{L}$, $\nabla_\lambda \mathcal{L}$, $\nabla_{\overline{\mu}} \mathcal{L}$, and $\nabla_{\underline{\mu}} \mathcal{L}$ are the partial gradients of $\mathcal{L}(x, \lambda, \overline{\mu}, \underline{\mu})$ with respect to x, λ, $\overline{\mu}$, and $\underline{\mu}$, respectively. The parameters K, Γ, $\Gamma_{\overline{\mu_i}}$, and $\Gamma_{\underline{\mu_i}}$ are diagonal matrices with ith diagonal components K_i, Γ_i, $\Gamma_{\overline{\mu}_i}$, and $\Gamma_{\underline{\mu}_i}$, respectively, and $[w]^+_z$ is an elementwise projection to the positive orthant defined as

$$[w]^+_z = \begin{cases} w; & w > 0 \text{ or } z > 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for all $w,\ z \in \mathbb{R}$.

The primal-dual dynamics (8) fall into the general gradient dynamics of (Arrow et al., 1958), and the stability and convergence properties for such dynamics have been investigated for congestion control (Feijer and Paganini, 2010) and for power optimization applications (Ma and Elia, 2013; Cherukuri et al., 2016). However, inspired by Costantini et al. (2008), we develop more compact primal-dual dynamics that can easily be implemented using a neural network architecture and with fast analog circuits (Levenson and Adegbege, 2016). We also construct sufficient conditions via Lyapunov stability to guarantee the asymptotic convergence of the primal-dual dynamics. Note that the stability proof of Costantini et al. (2008) is invalid for our case as the piecewise linear function corresponding to (1c) is not necessarily odd which is an underlying assumption in (Costantini et al., 2008).

3.1 Concise Primal-Dual Dynamics

To derive concise primal-dual dynamics for (1), we adopt the bound-constrained reformulation of the Lagrangian function (5) where only the equality constraint is incorporated as

$$\tilde{\mathcal{L}}(x, \lambda) = \frac{1}{2}x^T H(\lambda)x + q^T(\lambda)x + r(\lambda). \tag{10}$$

The bound constraint (1c) is enforced via the following quadratic programming (for fixed λ) sub-problem :

$$\underset{x}{\text{minimize}}\ \tilde{\mathcal{L}}(x, \lambda) \tag{11a}$$

$$\text{subject to } l \leq x \leq u. \tag{11b}$$

With this, the multipliers $\overline{\mu}$ and $\underline{\mu}$ have been eliminated resulting in a condensed reformulation for (5). Observe that the partial gradients of $\tilde{\mathcal{L}}(x, \lambda)$ with respect to x and λ are respectively,

$$\nabla_x \tilde{\mathcal{L}}(x, \lambda) = \nabla_x \mathcal{L}(x, \lambda, \overline{\mu}, \underline{\mu}) = H(\lambda)x + q(\lambda) \text{ and}$$

$$\nabla_{\lambda_i} \tilde{\mathcal{L}}(x, \lambda) = \nabla_{\lambda_i} \mathcal{L}(x, \lambda, \overline{\mu}, \underline{\mu}) = \frac{1}{2}x^T H_i x + q_i^T x + r_i.$$

The primal-dual dynamics for the condensed problem (11) can be implemented in two different forms. First we define the non-linearity

$$\sigma(x) = [\sigma_1(x_1) \ldots \sigma_n(x_n)]^T \tag{12}$$

where each $\sigma_i(x_i)$ is a pieceiwise linear function