Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-25 (2015) 271–276

On the Local Convexity of Singular Optimal Control Problems Associated with the Switched - Mode Dynamic Systems

V. Azhmyakov * **R. Juarez** ** **St. Pickl** ***

* *Faculty of Biomedical Engineering, Electronics and Mechatronics, Universidad Antonio Nariño, Neiva, Republic of Colombia (e-mail: vadim33tech@uan.edu.co)*
** *Manufacture Technologies Engeenieering, Universidad Politecnica of Gomez Palacio, Mexico (e-mail: rjuarez@upgop.edu.mx)*
*** *Institut für Theoretische Informatik, Mathematik und Operations Research, Fakultät für Informatik, Universität der Bundeswehr München, München, Germany (e-mail: stefan.pickl@unibw.de)*

Abstract: This paper studies a class of singular Optimal Control Problems (OCPs) governed by general switched - mode control processes. We develop a new constructive formalism for this class of switched dynamic systems and study OCPs with quadratic cost functionals. The initially given optimization problem is replaced by an auxiliary "weakly relaxed" OCP. The main aim of our contribution is with the formal proof of the local convexity property of the obtained auxiliary OCP. This fundamental convexity property makes it possible to apply powerful and relatively simple numerical optimization schemes to the initially given sophisticated singular OCP. The conceptual optimal control design framework we propose includes an optimal switching times selection ("optimal timing") as well as optimal modes scheduling ("optimal sequencing").

Keywords: switched dynamics, singular optimal control, infimal prox convolution, local convexity

1. INTRODUCTION

Constructive optimization of complex dynamic systems and the corresponding computer oriented methods and software are nowadays a usual and efficient design technology for practical development and improvement of several modern engineering systems (see e.g., [9, 14, 24, 26, 36]). Our paper studies mathematical aspects of a specific class of switched control systems, namely, so called switched - mode dynamic models (see [10, 26, 37-39]). We establish the local convexity property of a broad class of the weakly relaxed OCPs involving switched - mode dynamics. This theoretic contribution makes it possible to apply some powerful but relatively simple numerical algorithms from convex programming to the initially given OCP with a sophisticated structure. Following the basic concepts from [37-39] we consider so called "non - autonomous" switched - mode dynamic models and the associated quadratic - type OCPs. Let us note that various switched - mode control systems and the related OCPs have been comprehensively studied in the past several years due to their engineering applications. Let us mention here some notable applications from the mobile robot technology, automotive control, telecommunications, networking control and bio - mathematics. An interesting application of the optimal switched - mode dynamics to the control systems powered by a specific (multiple) data sources was discussed in [30]. We also refer to [10] and to references therein for various examples of the optimized switched - mode systems.

The main aim of our contribution is with the strong theoretic foundation for a future numerical treatment of the OCPs associated with the general switched - mode control systems. The optimization approach we propose includes an optimal sequencing scheduling as well as a simultaneously implemented optimal timing (see [39] for exact concepts). Note that some previously proposed optimization schemes (for example, the "mode - insertion algorithm" [3, 22]) include two separate optimization steps with respect to the a timing / sequencing. Otherwise, the widely used "gradient - descent" methodology (see [26] and [37-39]) is not sufficiently extended by the necessary numerical consistency statements. In our paper, we propose a conceptually new optimization approach based on a combination of a specific relaxation scheme and the first-order minimization. And, it should be noted already at this point that the optimization scheme we propose can be effectively implemented in a concrete systems design phase. The convergence results for the first - order optimization algorithms realised here in the form of a projected gradient method (see Theorem 2 and Theorem 3) constitute a main contribution of our work.

The remainder of this paper is organized as follows: Section 2 introduces the analytic model of a switched - mode dynamic system and contains the main problem formulation. In Section 3 we study the infimal based prox convolution in the context of a singular OCP involving switched - mode dynamics. Applications of the convex infimal convolution technique to the initially given OCP make it possible to

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.11.099

obtain a weakly relaxed optimization problem. Section 4 contains a short discussion of the numerical aspects in connection with the obtained theoretical results. Section 5 summarizes our paper.

2. OPTIMIZATION OF SWITCHED - MODE CONTROL SYSTEMS

2.1 The System Modelling Framework

Let us start by introducing a constructive self - closed formalism for the switched - mode control systems initially studied in [3, 22, 26, 37-39]. We consider the general (so called "non - autonomous") case that includes a conventional control variable $v(t) \in U \subseteq \mathbb{R}^m$

$$\dot{x}(t) = \sum_{i=1}^{I} \beta_{[t_{i-1},t_i)}(t) \sum_{k=1}^{K} q_{k,[t_{i-1},t_i)}(t) f_k(v(t), x(t)) \quad \text{a.e. on } [0, t_f], \quad x(0) = x_0 \in \mathbb{R}^n. \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is a state vector, the control domain U is a compact set and functions $f_k : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, where $k = 1, ..., K \in \mathbb{N}$, are assumed to be continuously differentiable. By

$$\beta_{[t_{i-1},t_i)}(t) = \begin{cases} 1 & \text{if } t \in [t_{i-1}, t_i) \\ 0 & \text{otherwise.} \end{cases}$$

we denote the characteristic functions of the disjunct time intervals $[t_{i-1}, t_i)$. We put

$$t_0 = 0, \ t_I = t_f.$$

The admissible inputs $v(\cdot)$ in (1) are elements of the following control set:

$$\mathcal{U} := \{w(\cdot) \in \mathbb{L}^\infty([0, t_f], \mathbb{R}^m) \ : \ w(t) \in U \text{ a.e. on } [0, t_f]\}.$$

It means that the conventional controls in (1) are assumed to be bounded and measurable functions. Let I be a (not specified) natural number and $q_{k,[t_{i-1},t_i)}(t) \in \{0, \ 1\}$ for all $k = 1, ..., K$, $t \in [0, t_f]$ such that

$$\sum_{k=1}^{K} q_{k,[t_{i-1},t_i)}(t) = 1.$$

Let us also introduce a "joint vector field" $F(v, x)$ for the initial value problem (1) and a "sequencing control vector" $q_{[t_{i-1},t_i)}(t)$ related to $[t_{i-1}, t_i)$:

$$F(v, x) := \{f_1(v, x), ..., f_K(v, x)\},$$
$$q_{[t_{i-1},t_i)}(t) := (q_{1,[t_{i-1},t_i)}(t), ..., q_{K,[t_{i-1},t_i)}(t))^T.$$

We next use the simplified notation $q^i(t) \equiv q_{[t_{i-1},t_i)}(t)$. Note that every component $q_{k,[t_{i-1},t_i)}(t)$ of $q^i(t)$ "selects" a particular mode $f_k(x)$ in (1) considered on the time interval $[t_{i-1}, t_i)$. Hence the sequencing control vector (a vector function) $q^i(\cdot) \in \mathcal{Q}_K$,

$$\mathcal{Q}_K := \{\theta : [0, t_f] \to \{0, \ 1\}^K \ | \ \sum_{k=1}^{K} q_{k,[t_{i-1},t_i)}(t) = 1\}$$

represents the schedule of modes from $F(v, x)$.

System (1) is also associated with an a priory unknown sequence of switching times

$$\tau := \{t_1, ..., t_{I-1}\}.$$

Usually a real - world engineering system admits a maximal number of switchings determined on a finite time interval. In other words we assume the absence of the Zeno behaviour in system (1) and additionally suppose $I \leq I_{max} \in \mathbb{R}$. We now introduce a "timing control vector"

$$\beta(t) := (\beta_{[t_0,t_1)}(t), ..., \beta_{t_{I-1},t_I}(t))^T.$$

By $\mathcal{B}_I$ we next denote the set of all vectors (vector functions) $\beta(\cdot)$ such that the resulting time intervals $[t_{i-1}, t_i)$ are disjunct and moreover, $I \leq I_{max}$. We now are ready to describe the set of admissible switching - type control inputs for system (1). Consequently, the non - conventional control vector $u(t)$ for (1) includes the sequencing control and the timing control vectors determined above

$$u(\cdot) := (\beta^T(\cdot), q^T(\cdot))^T \in \mathcal{B}_I \bigotimes (\mathcal{Q}_K)^I,$$

where $q := \{q^1, ..., q^I\}$ is a family of sequencing controls for every time interval $[t_{i-1}, t_i)$. The full control vector is now defined as a pair

$$\nu(\cdot) := (u^T(\cdot), v^T(\cdot))^T \in \mathcal{B}_I \bigotimes (\mathcal{Q}_K)^I \bigotimes \mathcal{U},$$

that includes the switching - type control $u(\cdot)$ and the conventional system input $v(\cdot)$. We next assume that for every feasible control function $\nu(\cdot)$ the initial value problem (1) has an absolutely continuous solution $x^\nu(\cdot)$ (on the given time interval). However, due to the presence of the conventional and switched - type nonlinearities in the above equation, the global existence of a solution to (1) is not guaranteed, unless there are some additional conditions imposed. We refer to [17, 23] for the corresponding details. Let us also note that the boundedness of the control vector $\nu(\cdot)$ implies the boundedness (almost everywhere) of the derivatives of $x^\nu(\cdot)$. Using the general results from [23, 25], the analytic technique we propose can also be extended to the dynamic systems (1) equipped with the unbounded admissible control functions and as a consequence with the corresponding (general) absolutely continuous trajectories.

Taking the given vector field $F(v, x)$ into consideration, we can rewrite system (1) in the compact form

$$\dot{x}(t) = \langle \ \beta(t), \ \left(\langle q^i(t), F(v(t), x(t))\rangle_K\right)_{i=1,...,I} \ \rangle_I \tag{2}$$

where $\langle \cdot, \cdot \rangle_N$ denotes a scalar product in the N-dimensional Euclidean space. Note that equation (2) constitutes a conventional dynamic system with the smooth right hand side. Clearly, in the absence of the conventional control variable $v(\cdot)$ system (2) represents a "system with piecewise constant inputs" from [8, 9].

2.2 Singular OCPs with Switched - Mode Dynamics

Given a switched - mode system (1) we now formulate the main OCP

$$\begin{aligned} &\text{minimize } J(\nu(\cdot)) \\ &\text{subject to (2)}, \ \nu(\cdot) \in \mathcal{B}_I \bigotimes (\mathcal{Q}_K)^I \bigotimes \mathcal{U}, \end{aligned} \tag{3}$$

with the quadratic cost functional

$$J(\nu(\cdot)) := \frac{1}{2} \int_{t_0}^{t_f} \left(\langle Q(t)x(t), x(t)\rangle + \langle R(t)v(t), v(t)\rangle\right) dt + \frac{1}{2}\langle Gx(t_f), x(t_f)\rangle,$$

where $G \in \mathbb{R}^{n \times n}$ a symmetric positive defined matrix and $Q(\cdot)$, $R(\cdot)$ are integrable matrix-functions that satisfy standard symmetry and positivity hypothesis

$$Q(t) \geq 0, \ R(t) \geq \delta I, \ \delta > 0 \ \ \forall \ t \in [t_0, t_f].$$