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Optimal tracking and disturbance rejection with invariant zeros on the unit circle: a polynomial spectral factorization design

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Abstract: We present a simple algorithm for computation of $\mathscr{H}_2$-optimal tracking and disturbance rejection controller of discrete-time systems possessing invariant zeros on the unit circle, based on polynomial spectral factorization. We prove that the column degrees of the associated para-hermitian polynomial matrix to be factorized are equal to the plant controllability indices. A numerical/computer simulation example is given.

Keywords: Tracking, disturbance rejection, para-hermitian matrices, polynomial J-spectral factorization, optimal LQ return difference equality.

1. INTRODUCTION

Tracking and/or disturbance rejection is one of the most important control problems. There are various approaches in the literature (Chen et al (2002), Circa et al (2005), Hoover et at (2004), Leva & Bascetta (2007), Tsai et al (2014), Wang et al (2011), Willems & Mareels (2004), Xie et al (2000)). In Mosca (1995), Chapter 4 elaborates on the spectral factorization method for LQ control problem (which is less general than our problem of LQ tracking with disturbance rejection). It can also be approached by the *exact output regulation* result (see Section 2.3 in Saberi et al. (2000)). For that purpose, define an antistable exogenous system:

$$x_1(k+1) = A_{11}x_1(k) \ , \tag{1}$$

and plant with vector inputs $u(k)$, m-dimensional, and $x_1(k)$, n_1-dimensional, and vector output $h(k)$:

$$\left.\begin{aligned} x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k) \\ h(k) &= C_1x_1(k) + C_2x_2(k) + Du(k) \end{aligned}\right\} \ , \tag{2}$$

where (A_{22}, B_2) is a stabilizable pair, $x_2(k)$ is n_2-dimensional state vector, and consider the problem of minimization of the criterion

$$\mathcal{J} = \textstyle\sum_{k=0}^{\infty} h^{\mathrm{T}}(k)h(k) \ . \tag{3}$$

It is shown on page 196 of Trentelman et al. (2001) (see also Example 1) that the system (1) and (2) can be applied for tracking and disturbance rejection, by considering that the antistable dynamics of the reference (to-be-tracked) signal and disturbance is actually the exogenous system (1).
It is obvious that the system (1) and (2) is unstabilizable, and it belongs to the category of *non-autonomous external- and internal-stationary system* (see the categorization of systems at Dimirovski et at (1977)).

It is known (Theorem 2.3.1 Saberi et al. (2000)) that if the system (A_{22}, B_2, C_2, D) has no invariant zeros on the unit circle, i.e. matrix pencil

$$\begin{bmatrix} zI - A_{22} & -B_2 \\ C_2 & D \end{bmatrix} \tag{4}$$

has no finite generalized eigenvalues (FGEs) on the unit circle, there is a control $u(k)$ such that $h(k) \to 0$ ($k \to \infty$) for all initial conditions $x_1(0)$, $x_2(0)$ (and consequently the criterion (3) can be finite), and the closed loop system without the exogenous inputs is stable, if and only if there is a solution (V, K_3) of the following matrix system:

$$\left.\begin{aligned} A_{22}V - VA_{11} &= A_{21} - B_2K_3 \\ C_1 - C_2V &= DK_3 \end{aligned}\right\} \ . \tag{5}$$

Note that the condition (5) is satisfied generically (*well-posed problem* (see Section 9.2 in Trentelman et al. (2001))) only when the transfer matrix from the control input to the output is right-invertible (see Theorem 9.10 in Trentelman et al. (2001)).

The plants with invariant zeros on the boundary of the stability region frequently appear in control practice. In that case, the requirement that "the closed loop system without the exogenous inputs is stable" of Saberi et al. (2000) cannot be satisfied simultaneously with minimality of the criterion (3). However, this requirement is too strong, because the system never works without the ex-

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ogenous inputs. It is stated in Stefanovski (2015) that this requirement can be replaced with the requirement that the unstable modes of the closed-loop system are at most the unstable eigenvalues of the matrix A_{11}. Then, instead of the condition (5), it is introduced the more general condition

$$\left.\begin{array}{l} A_{22}V - VA_{11} = A_{21} - B_2K_3 + W_1M \\ C_1 - C_2V = DK_3 \end{array}\right\} , \quad (6)$$

for some matrices V, K_3 and M, where W_1 is a matrix with orthonormal columns which span the right kernel of matrix Y_{22}, ($W_1 = \mathrm{null}(Y_{22})$) where (Y_{22}, K_2) is a marginally stabilizing solution of the DARS

$$\begin{array}{l} Y_{22} = A_{22}^{\mathrm{T}}Y_{22}A_{22} + C_2^{\mathrm{T}}C_2 - \left(B_2^{\mathrm{T}}Y_{22}A_{22} + D^{\mathrm{T}}C_2\right)^{\mathrm{T}} K_2 \\ \left(B_2^{\mathrm{T}}Y_{22}B_2 + D^{\mathrm{T}}D\right) K_2 = B_2^{\mathrm{T}}Y_{22}A_{22} + D^{\mathrm{T}}C_2 \ . \end{array} \quad (7)$$

Under the following assumption:

Assumption 1. The FGEs of matrix pencil $I - z(A_{22} - B_2K_2)^{\mathrm{T}}$ in $|z| > 1$ pairwise differ of the eigenvalues of matrix A_{11}

it is proved in Theorem 5 of Stefanovski (2015) that a minimizing control of the criterion (3) is given by $u(k) = -K_1x_1(k) - K_2x_2(k)$, $K_1 = K_3 + K_2V$.

Consider "two-sided" polynomial matrix $\boldsymbol{P}(z)$,

$$\boldsymbol{P}(z) = P_\nu^{\mathrm{T}}z^{-\nu} + \cdots + P_1^{\mathrm{T}}z^{-1} + P_0 + zP_1 + \cdots + z^\nu P_\nu \quad (8)$$

for some real $m \times m$ -dimensional matrices $P_0, P_1, \ldots, P_\nu$, such that $P_0^{\mathrm{T}} = P_0$. The polynomial matrix $\boldsymbol{P}(z)$ is para-hermitian, i.e. $\boldsymbol{P}^{\mathrm{T}}(z^{-1}) = \boldsymbol{P}(z)$. The zeros of a para-hermitian matrix are distributed symmetrically in respect to the unit circle $|z| = 1$. A problem of polynomial J-spectral factorization is to find a nonsingular polynomial matrix $\boldsymbol{\Phi}(z)$ such that

$$\boldsymbol{P}(z) = \boldsymbol{\Phi}^{\mathrm{T}}(z^{-1})J\boldsymbol{\Phi}(z) \ , \quad (9)$$

and the zeros of $\boldsymbol{\Phi}$ are in $|z| \leq 1$, where J is a signature matrix: $J = \mathrm{diag}\{-I_{m_-},\ 0_{m_0\times m_0},\ I_{m_+}\}$, for some indices m_-, m_0 and m_+ such that $m_- + m_0 + m_+ = m$.

The standard result on existence of J-spectral factorization (9) (see Trentelman & Rapisarda (1999) for continuous-time J-spectral factorization) states that it exists if and only if $\boldsymbol{P}$ has constant signature everywhere on the unit circle, except on its zeros.

The polynomial spectral factorization method has been applied for LQ control (which is slightly more general than $\mathscr{H}_2$ control) in the literature: For continuous-time LQ control via spectral factorization, we mention the works Kučera et al. (1999) and Willems (1993). The algorithms of these works can be adapted for discrete-time LQ control. A drawback of these algorithms is their complexity. In particular:

In Kučera et al. (1999), LQ control is achieved by obtaining a matrix fraction description, a spectral factorization, and solving a Diophantine equation for a constant matrix.

In Willems (1993), LQ control is achieved by solving a quadratic polynomial matrix equation.

In this paper we present an $\mathscr{H}_2$ optimal state-feedback control for simultaneous tracking and disturbance rejection of plants possessing invariant zeros on the unit circle. As a technical method, we use the polynomial spectral factorization.

The paper organization is as follows. In Section 2 we present an algorithm for J-spectral factorization. By reversing arguments, in Section 3, we develop an algorithm for $\mathscr{H}_2$-optimal control (tracking and/or disturbance rejection) based on spectral factorization of polynomial matrices. Besides one polynomial spectral factorization, it requires a canonical decomposition of the given control system. A numerical example with invariant zeros of the plant is given, to verify the performance of the proposed controller.

Remarks on the notation. By the superscript T we denote matrix transposition. The identity matrix is denoted by I, or I_n if the matrix dimension is requiring. The matrix functions of z we write bold-faced. If $\boldsymbol{W}(z)$ is a matrix function, by $\boldsymbol{W}^{\#}$ we denote the matrix function $\boldsymbol{W}^{\mathrm{T}}(z^{-1})$. By $\boldsymbol{W}^{-\#}$ we denote the matrix function $\left(\boldsymbol{W}^{\mathrm{T}}(z^{-1})\right)^{-1}$. The zeros of a rational matrix (and also of $\boldsymbol{P}(z)$ (8)) are defined by its McMillan form. The abbreviation DARS means discrete-time algebraic Riccati system, and the abbreviation FGE means finite generalized eigenvalue. ■

2. AN ALGORITHM FOR POLYNOMIAL J-SPECTRAL FACTORIZATION

Before we proceed with the formulation of the polynomial J-spectral factorization algorithm from Stefanovski (2004), at first we formulate the discrete version of optimal LQ return difference equality, Ionescu et al. (1997), stated in Proposition 1.

Proposition 1. (Discrete optimal LQ return difference equality) For some matrices A, B and symmetric Σ with compatible partition

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} ,$$

let there exists a marginally stabilizing (in the sense below) solution (X, F) of the following DARS

$$\begin{array}{l} X = A^{\mathrm{T}}XA + \Sigma_{11} - \left(B^{\mathrm{T}}XA + \Sigma_{21}\right)^{\mathrm{T}} F \\ \left(B^{\mathrm{T}}XB + \Sigma_{22}\right) F = B^{\mathrm{T}}XA + \Sigma_{21} \end{array} \quad (10)$$

Then for almost all complex numbers z, the following identity holds true

$$\begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}^{\#} \Sigma \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}$$
$$= \left[I + F(zI - A)^{-1}B\right]^{\#} \left(B^{\mathrm{T}}XB + \Sigma_{22}\right) \left[I + F(zI - A)^{-1}B\right] . \quad (11)$$

By marginally stabilizing solution of (10) we mean a solution such that the eigenvalues of the matrix $A - BF$ lie in the region $|z| \leq 1$.

In the most of applications (see Kwakernaak & Šebek (1994), Trentelman & Rapisarda (1999)), the matrix $\boldsymbol{P}(z)$ is given in the pre-factorized form

$$\boldsymbol{P}(z) = \Xi^{\mathrm{T}}(z^{-1})Q\Xi(z) \ , \quad (12)$$

for some constant symmetric matrix Q and polynomial matrix $\Xi(z)$, in which the number of rows is not less than the number of columns m. In particular, as elaborated in Section 3, $\mathscr{H}_2$-optimal control problem leads to such a decomposed polynomial matrix. For such a given polynomial matrix $\boldsymbol{P}(z)$, we can avoid the result contained in the following Proposition 3.