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Inverse Optimal Control Problem: the Sub-Riemannian Case ⋆

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Abstract: The object of this paper is to study the uniqueness of solutions of inverse control problems in the case where the dynamics is given by a control-affine system without drift and the costs are length and energy functionals.

Keywords: Optimal control, sub-Riemannian geometry, optimal trajectories, geodesics, inverse problem, nonholonomic systems, projective equivalence, affine equivalence.

1. INTRODUCTION

This paper is motivated by recent applications of optimal control theory to the study of human motions. Indeed, it is a widely accepted opinion in the neurophysiology community that human movements follow a decision that undergoes an optimality criterion (see Todorov (2006)). Finding this criterion amounts to solve what is called an *inverse optimal control problem*: given a set Γ of trajectories (obtained experimentally) and a class of optimal control problems – that is, a pair (control system, class $\mathcal{C}$ of costs) – suitable to model the system, identify a cost function φ in $\mathcal{C}$ such that the elements of Γ are minimizing trajectories of the optimal control problem associated with φ. Note that we restrict ourselves to integral costs, so the class $\mathcal{C}$ is actually the class of the infinitesimal costs.

The first two main aspects in the inverse optimal control problem are the question of existence of such an infinitesimal cost φ in the class $\mathcal{C}$, and the question of its uniqueness in this class. The existence part, even within the problems in classical Calculus of Variation, where $\mathcal{C}$ is the set of all smooth Lagrangians, is still an open problem, which attracted a lot of attention since the creation of Calculus of Variation (see a survey in Saunders (2010)). In the present paper the existence is assumed to hold a priori and the main question is the uniqueness of the cost φ in the class $\mathcal{C}$ or generically in the class $\mathcal{C}$, up to a multiplication by a positive constant.

It is easy to construct examples where the uniqueness does not hold. If the set Γ consists of unparameterized straight lines in $\mathbb{R}^2$, then in the class of length functionals with respect to Riemannian metrics on $\mathbb{R}^2$, there are functionals corresponding to Riemannian metrics with nonzero Gaussian curvature having Γ as their geodesics (see the example in subsection 3.1), so these functionals are not constantly proportional to the Euclidean length functional. Note also that by a classical theorem by Beltrami (1869), these functionals are the only ones with such property within this class. If one extends the class of functionals to Lagrangians, then one arrives to the variational version of Hilbert's fourth problem in dimension 2, which was solved by Hamel (1903), and provides a very rich class of Lagrangians having straight lines as extremals.

These examples are related to functionals without dynamical constraints, i.e. for which the space of admissible curves is defined by a trivial control system $\dot{x} = u$. If we consider the simplest class of optimal control problems, the linear-quadratic ones (the control system is linear and the cost is quadratic w.r.t. both state and control), the cost can be explicitly reconstructed from the optimal trajectories at least in the mono-input case, see Nori and Frezza (2004) and Berret and Jean (2016).

The present paper is devoted to the inverse problem for optimal control problems with a dynamical constraint given by a control-affine systems without drift and with two classes of functionals: the energy functionals (i.e. where the infinitesimal cost is quadratic with respect to control) and the length functionals (where the infinitesimal cost is just the square root of the infinitesimal energy cost). The first class of these optimal control problems (i.e. with the energy functionals) can be seen as a generalization of the class of linear-quadratic problems to the same extend as the energy functionals with respect to an arbitrary Riemannian metrics are generalizations of the corresponding Euclidean ones.

These two kinds of inverse problems can be reformulated in more geometric terms as problems of affine and projective equivalence of sub-Riemannian metrics, which in the case of Riemannian metrics are both classical: the classification of locally projectively equivalent Riemannian metrics under some natural regularity assumptions was done by Levi-Civita (1896) as an extension of the result of Dini

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(1870) for surfaces. The affinely equivalent Riemannian metrics are exactly the metrics with the same Levi-Civita connection and the description of the pairs of Riemannian metrics with this property can be attributed to Eisenhart (1923). The only complete classification of projectively equivalent metrics in a proper sub-Riemannian case was done far more recently in Zelenko (2006) for contact and quasi-contact sub-Riemannian metrics.

The paper is organized as follows. We first detail in section 2 the different notions of equivalence between infinitesimal costs and between metrics and show how they are related to the uniqueness of solutions of the corresponding inverse optimal control problems. We then expose in section 3 the results on equivalence of metrics in the Riemannian, contact and quasi-contact cases and their consequences for inverse problems. We adopt in this exposition the unifying point of view of the generalized Levi-Civita pairs and propose a general conjecture for the classification of affinely and projectively equivalent metrics. Finally we announce in section 4 our results on the equivalence of general sub-Riemannian metrics, showing in particular that for generic distributions all metrics are affinely rigid.

2. REDUCTION TO PROJECTIVE AND AFFINE EQUIVALENCE OF SUB-RIEMANNIAN METRICS

Let M be an n-dimensional smooth and connected manifold. Given a control system $\dot{q} = f(q, u)$ on M with a control space U we assign to any smooth infinitesimal cost $\varphi(q, u)$ the following family of optimal control problems parameterized by the initial and terminal times $a < b$ and by the initial and terminal points q_0, q_1:

$$\int_a^b \varphi(q, u)dt \to \min, \tag{1}$$
$$\dot{q} = f(q, u),$$
$$q(a) = q_0, q(b) = q_1.$$

Note that, since the dynamical constraint and the cost are autonomous, by a time translation one can always make $a = 0$, but we prefer not to do it in order not to have unnecessary restrictions on possible time-parameterizations of minimal trajectories.

Definition 1. We say that two infinitesimal costs φ and $\widetilde{\varphi}$ are equivalent via minimizers if the corresponding families of optimal control problems have the same minimizing trajectories.

It is clear that, in a given class $\mathcal{C}$, the existence of two distinct infinitesimal costs which are equivalent via minimizers implies that the inverse optimal control problem does not have uniqueness property in this class.

The set of minimizers is in general not easy to handle, it is easier to work with the extremals of (1). Recall that an *extremal trajectory* of (1) is a trajectory satisfying the conditions of the Pontryagin Maximum Principle, i.e. it is the projection q of a curve (q, p) on T^*M solution of some Hamiltonian equations arising from the maximisation w.r.t. u of $H(p, q, u, p^0) = \langle p, f(q, u)\rangle + p^0\varphi(q, u)$, where $p^0 \leq 0$ is a scalar. Every minimizer is an extremal trajectory. This suggests a second notion of equivalence.

Definition 2. We say that two infinitesimal costs φ and $\widetilde{\varphi}$ are equivalent via extremal trajectories if the corresponding families of optimal control problems have the same extremal trajectories.

Both notions of equivalence are different in general, but we will see below that in particular cases the first one implies the second.

We consider now a control-affine system without drift,

$$\dot{q} = \sum_{i=1}^{m} u_i X_i(q), \qquad q \in M, \tag{2}$$

where $X_1, \ldots, X_m$ are vector fields on M and the control $u = (u_1, \ldots, u_m)$ takes values in $\mathbb{R}^m$. We assume that the Lie algebra generated by the vector fields $X_1, \ldots, X_m$ is of full rank, i.e. $\dim \mathrm{Lie}(X_1, \ldots, X_m)(q) = n$ for every $q \in M$, which guarantees that the system is controllable. Such a system is called a *Lie bracket generating nonholonomic system*. We make the additional assumption that $D(q) = \mathrm{span}\{X_1(q), \ldots, X_m(q)\}$ is of constant rank equal to m, which implies that D defines a rank m distribution (i.e. a rank m subbundle of TM), $X_1, \ldots, X_m$ being *a frame* of the distribution. Note that we can always make this assumption in a neighbourhood of a generic point (up to reducing m).

Define $\mathcal{C}$ as the set of smooth functions $g : M \times \mathbb{R}^m \to \mathbb{R}$, $(q, u) \mapsto g(q)(u)$, such that for every $q \in M$, $g(q)(\cdot)$ is a positive definite quadratic form. From a more geometric viewpoint, we can see g as a function on D and write $g(\dot{q})$ instead of $g(q)(u)$ for $\dot{q}$ satisfying (2). Thus the set $\mathcal{C}$ appears as the set of the sub-Riemannian metrics on (M, D) and, in the particular case where $m = n$ (and so $D = TM$), $\mathcal{C}$ is the set of the Riemannian metrics on M. Any $g \in \mathcal{C}$ is the infinitesimal cost for the *energy functional*, while $\sqrt{g}$ is the infinitesimal cost for the *length functional* associated with the sub-Riemannian metric g.

Since two constantly proportional metrics define the same energy and length minimizers, the problem of injectivity can be stated as follows.

Inverse sub-Riemannian problems *Let M be a manifold and D a distribution on M. Can we recover g in a unique way, up to a multiplicative constant, from the knowledge of all energy minimizers of (M, D, g)? And from the knowledge of all length minimizers of (M, D, g)?*

When the answer to one of the above questions is positive, we say that the corresponding inverse sub-Riemannian problem for (M, D) is injective.

Now let us try to characterize the injectivity of the above problem through equivalence via extremal trajectories. Given a sub-Riemannian metric g on (M, D), the extremal trajectories of the energy functional are called the *sub-Riemannian geodesics*. There are two type of sub-Riemannian geodesics, normal and abnormal (see Montgomery (2002) or Rifford (2014) for details). The alternative is not exclusive, a geodesic can be both normal and abnormal. If it is not the case we will say that the geodesic is either strictly normal or strictly abnormal. In the Riemannian case (i.e. $D = TM$) there are no abnormal geodesics and the normal geodesics coincide with the usual