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# Necessary conditions for a nonclassical control problem with state constraints ⋆

**Chems Eddine Arroud** * **Giovanni Colombo** **

* *Department of Mathematics, Jijel University, Jijel, Algeria, and Mila University Center, Mila, Algeria (e-mail: arroud.math@gmail.com)*
** *Dipartimento di Matematica "Tullio Levi-Civita", University of Padova, Padova, Italy (e-mail: colombo@math.unipd.it).*

**Abstract:** We consider the problem of minimizing the cost $h(x(T))$ at the endpoint of a trajectory $x$ subject to the finite dimensional dynamics

$$\dot{x} \in -N_C(x) + f(x,u), \quad x(0) = x_0,$$

where $N_C$ denotes the normal cone to the convex set $C$. Such differential inclusion is termed, after Moreau, *sweeping process*. We label it as a "nonclassical" control problem with state constraints, because the right hand side is discontinuous with respect to the state, and the constraint $x(t) \in C$ for all $t$ is implicitly contained in the dynamics.
We prove necessary optimality conditions in the form of Pontryagin Maximum Principle by requiring, essentially, that $C$ is independent of time. If the reference trajectory is in the interior of $C$, necessary conditions coincide with the usual ones. In the general case, the adjoint vector is a BV function and a signed vector measure appears in the adjoint equation.



*Keywords:* Moreau's sweeping process, Optimal control, Pontryagin Maximum Principle.

## 1. INTRODUCTION

The *sweeping process* was introduced by Moreau in the Seventies as a model for dry friction and plasticity (see Moreau (1974)) and later studied by several authors. In its perturbed version, it features the differential inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t)), \quad t \in [0,T] \tag{1}$$

coupled with the initial condition

$$x(0) = x_0 \in C(0). \tag{2}$$

Here $C(t)$ is a closed moving set, with normal cone $N_{C(t)}(x)$ at $x \in C(t)$. The space variable, in this paper, belongs to $\mathbb{R}^n$. If $C(t)$ is convex, or mildly non-convex (in a sense that will not be made precise here), and is Lipschitz as a set-valued map depending on $t$, and the perturbation $f$ is Lipschitz as well, then it is well known that the Cauchy problem (1), (2) admits one and only one Lipschitz solution (see, e.g., Thibault (2003)). Observe that the state constraint $x(t) \in C(t)$ for all $t \in [0,T]$ is *built in the dynamics*, being $N_{C(t)}(x)$ empty if $x \notin C(t)$: should a solution $x(\cdot)$ exist, then automatically $x(t) \in C(t)$ for all $t$. If a control parameter $u$ appears within $f$, then one is lead to study problems of the type

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t),u(t)), \quad u(t) \in U \tag{3}$$

subject to (2), aiming, for example, at

$$\text{minimizing } h(x(T)), \tag{4}$$

the final cost $h$ being smooth. There is a clear difference with classical control problems with state constraints (see, e.g., Vinter (2000)), where the constraint does not appear explicitly in the dynamics: in this case the right hand side of the dynamics is not Lipschitz with respect to the state variable, but indeed has only closed graph. This fact is a source of major difficulties in deriving necessary optimality conditions for (3), (4).

In recent years (see, e.g., Bagagiolo (2002), Gudovich et al. (2011), Brokate et al. (2013), Colombo et al. (2016), Colombo et al. (2016), Arroud et al. (2016), and Cao et al. (2017), and references therein) some papers dealing with control problems involving the sweeping process were published, the control appearing in the perturbation $f$ and/or in the moving set $C$. Several necessary conditions were established, under different kinds of assumptions, or a Hamilton-Jacobi characterization of value function was proved. The present paper is devoted to prove a result inspired by Arroud et al. (2016) and Brokate et al. (2013). More precisely, we prove necessary conditions of Pontryagin maximum principle type for (4) subject to (3) and (2), the control appearing only within $f$, in the case where $C(\cdot)$ is constant, smooth and convex (see Theorems 2 and 3). The case where $C$ satisfies milder convexity assumptions and is not necessarily constant was treated in Arroud et al. (2016) with an extra assumption, while Brokate et al. (2013) contains results for a particular control problem involving a fixed smooth and *uniformly convex* set $C$. More preccisely, differently from Colombo et al. (2016) and Cao et al. (2017), where discrete approximations are used, in both Brokate et al. (2013) and Arroud et al. (2016) the authors use a penalization technique. The classical Moreau-Yosida regularization allows in Arroud et al.

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(2016) to relax the uniform convexity assumption, at the price of requiring a strong *outward pointing condition* on $f$ in order to treat the discontinuity of second derivatives of the squared distance function at the boundary of $C(t)$. In Brokate et al. (2013), the authors adopt a suitable smoothing of the distance, which on one hand needs $C(t)$ constant and uniformly convex and $0 \in C$, while on the other avoids imposing further compatibility assumptions between $f$ and $C$. In this paper we adapt to our situation the method developed in Brokate et al. (2013) and remove the assumption of strict convexity on $C$. The main technical part is Section 4.

## 2. PRELIMINARIES AND ASSUMPTIONS

**Notation.** We define the distance from a set $C \subset \mathbb{R}^n$ as $d(x) = \inf\{\|y-x\| : y \in C\}$ and *signed* distance from $C$ as $d_S(x) = d(x)$ if $x \notin C$ and $d_S(x) = -\inf\{\|y-x\| : y \notin C\}$ if $c \in C$. The normal cone to a convex set $C$ is defined as $N_C(x) = \emptyset$ if $x \notin C$ and $N_C(x) = \{v \in \mathbb{R}^n : \langle v, y-x\rangle \le 0 \ \forall y \in C\}$ if $x \in C$.

**Assumptions on the set $C$.** Let

$$C = \{x \in \mathbb{R}^n : g(x) \le 0\},$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^2$ with gradient $\nabla g \neq 0$ on the boundary $\partial C$ of $C$, and with the Hessian matrix $\nabla^2 g(x)$ positive semidefinite for all $x \in \mathbb{R}^n$. Assume furthermore that $g(\cdot)$ is coercive, so that $C$ is compact (and convex) and that $g(0) < 0$, so that $0 \in C$ and $C$ has nonempty interior. Observe that under our assumption the signed distance $d_S(x)$ from $C$ is of class $\mathcal{C}^2$ in a neighborhood of $\partial C$.

**Assumptions on the dynamics and the cost.** The control set $U \subset \mathbb{R}^n$ is compact and $f$ is continuous and bounded, say by a constant $\beta$, and is of class $\mathcal{C}^1$ with respect to $x$, with $\|\nabla_x f(x,u)\| \le L$ for all $x, u$. The cost $h$ is smooth.

Let now $\psi(x)$ be a $\mathcal{C}^2$ smoothing of $d_S$ in the interior of $C$ (which is $< 0$ in $\operatorname{int} C$ and is such that $\nabla\psi(x)$ is the unit external normal to $C$ at $x$ for every $x \in \partial C$). Set also

$$\Psi(x) = \frac{1}{3}\psi^3(x)\, 1_{(0,+\infty)}(\psi(x)).$$

Observe that $\Psi(\cdot)$ is of class $\mathcal{C}^2$ and convex in the whole of $\mathbb{R}^n$ and that both $\nabla\Psi(\cdot)$ and $\nabla^2\Psi(\cdot)$ vanish on $C$. Moreover one has

$$d(x)\nabla\psi(x) = d(x)\nabla d(x), \quad (5)$$

$$\nabla\Psi(x) = d^2(x)\nabla d(x), \quad (6)$$

$$\nabla^2\Psi(x) = 2d(x)\nabla d(x) \otimes \nabla d(x) + d^2(x)\nabla^2 d(x), \quad (7)$$

because in $C$, and in particular at the points where $\nabla d(x)$ does not exist (namely, in $\partial C$), both sides of the above expressions vanish, and outside $C$ they coincide.

## 3. THE REGULARIZED PROBLEM

Consider the regularized dynamics

$$\dot{x}(t) = \frac{-1}{\varepsilon}\nabla\Psi(x(t)) + f(x(t), u(t)),\ x(0) = x_0, \quad (8)$$

where $\varepsilon > 0$ and $u(t) \in U$ for all $t$. For each given $u$, this Cauchy problem admits a unique solution $x_\varepsilon$ for each $\varepsilon > 0$ on a maximal interval of existence. It is not difficult to prove that this interval is $[0,T]$ (see the proof of Proposition 1).

For every $\varepsilon > 0$ and every *global* minimizer $x_*, u_*$ of (4) subject to (3) and (2), we consider the *approximate problem* $P_\varepsilon(u_*)$

$$\text{minimize } h(x(T)) + \frac{1}{2}\int_0^T \|u(t) - u_*(t)\|^2\, dt, \quad (9)$$

over controls $u$, where $x$ is a solution of (8). By standard results, $P_\varepsilon(u_*)$ admits a global minimizer $u_\varepsilon$, with the corresponding solution $x_\varepsilon$. Necessary conditions of the original problem will be obtained by passing to the limit along conditions for $P_\varepsilon(u_*)$.

### 3.1 *A priori estimates for the regularized problem*

*Proposition 1.* Let $\varepsilon_n \to 0$ and let $(u_n, x_n)$ be a solution of the problem $P_{\varepsilon_n}$. Then, up to a subsequence, $u_n$ converges strongly in $L^2(0,T)$ to $u_*$ and $x_n$ converges weakly in $W^{1,2}(0,T)$ to $x_*$.

**Proof.** Since $0 \in C$ and so $\nabla\Psi(0) = 0$, by the convexity of $\Psi$ we obtain that $\langle\nabla\Psi(x), x\rangle \ge 0$ for all $(x)$. Thus

$$\|x_n(t)\| - \|x_0\| =$$

$$= \int_0^t \Big\langle \frac{x_n(t)}{\|x_n(t)\|}, \frac{-1}{\varepsilon_n}\nabla\Psi(x_n(t)) + f(x_n(t), u_n(t))\Big\rangle\, dt \le \beta,$$

which, in particular, implies that $x_n$ is defined in the whole of $[0,T]$. Moreover,

$$\|\dot{x}_n\|^2_{L^2} \le \int_0^T \Big\langle \dot{x}_n(t), \frac{-1}{\varepsilon_n}\nabla\Psi(x_n(t)) + f(x_n(t), u_n(t))\Big\rangle\, dt$$

$$= \int_0^T \Big(\frac{-1}{\varepsilon_n}\frac{d}{dt}\Psi(x_n(t)) + \langle f(x_n(t), u_n(t)), \dot{x}_n(t)\rangle\Big)\, dt$$

$$= \frac{-1}{\varepsilon_n}\Psi(x_n(T)) + \frac{1}{\varepsilon_n}\Psi(0) + \beta\int_0^T \|\dot{x}_n(t)\|\, dt$$

$$\le \beta\sqrt{T}\|\dot{x}_n\|_{L^2},$$

where we have used the fact that $0 \in C$ and that $\psi(x_n(T)) \ge 0$. The above estimate implies that the sequence $\dot{x}_n$ is uniformly bounded in $L^2(0,T)$. Thus, up to a subsequence, $x_n$ converges weakly in $W^{1,2}(0,T)$ to $\bar{x}$. Observe now that the uniform boundedness of $\|\dot{x}_n\|_{L^2(0,T)}$ together with (6), recalling (5) imply that

$$\|d(x_n(\cdot))^2\|_{L^2(0,T)} \le K\varepsilon_n \quad (10)$$

for a suitable constant $K$. Thus $x(t) \in C$ for all $t$. Again up to a subsequence, $u_n$ converges weakly in $L^2(0,T)$ to some $\bar{u}$. By using the very same argument of Proposition 4.3 in Arroud et al. (2016), one can prove that $\bar{x}$ is the